Analysis II

Midterm Exam

March 18, 2019

1. Show that, as $x \to \infty$,
$$\int_x^\infty \frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x \sim \frac{\mathrm{e}^{-x}}{x}\,.$$
(5)

2. (a) Let A be an open subset and B be a closed subset of a metric space. Determine whether the following occur always, sometimes, or never; give proofs or (counter)-examples.
   (i) $A \setminus B \equiv \{x \in A \colon x \notin B\}$ is open.
   (ii) $A \setminus B$ is closed.
   (iii) $A \setminus B$ is both open and closed.
   (iv) $A \setminus B$ is neither open nor closed.

   (b) Let X, Y, and Z be metric spaces, and $f \colon Y \to Z$ and $g \colon X \to Y$ be continuous mappings. Use the topological characterization of continuity to show that the composition $f \circ g$ is continuous.

(5+5)

3. (a) Let $\{x_n\}$ be a converging sequence of elements in a metric space X with limit x. Show that the set
$$E = \{x\} \cup \{x_n \colon n \in \mathbb{N}\}$$
is compact.

   (b) Prove that the boundary of a compact set is compact.

(5+5)

4. Let $f_n, g_n \colon \mathbb{R} \to \mathbb{R}$ be sequences of bounded continuous functions which converge uniformly to functions f and g, respectively.

   (a) Show that $f_n\, g_n \to f\, g$ uniformly as $n \to \infty$.

   (b) Give an example which shows that convergence may fail to be uniform when the assumption of boundedness is dropped.

(5+5)

5. Consider the sequence of functions on $[0, 1]$ defined by

$$f_n(x) = n \sin \frac{x}{n} .$$

   (a) Show that there exists a uniformly converging subsequence of $\{f_n\}$.
   (b) What is the limit function?

(5+5)

6. Let $f \in C([0, 1])$. Show that there exists a sequence of polynomials $\{p_n\}$ satisfying $p_n(0) = f(0)$ such that $p_n \to f$ uniformly on $[0, 1]$. (5)

7. Find a power series expansion centered at 0 for

$$\ln(1 + x)$$

and determine the radius of convergence. (5)