1. Show that, as $x \to \infty$,

$$\int_x^\infty \frac{e^{-\tilde{x}}}{\tilde{x}}\, d\tilde{x} \sim \frac{e^{-x}}{x}.$$

(5)

$$\int_x^\infty \frac{e^{-t}}{t}\, dt = -\frac{e^{-t}}{t}\Big|_x^\infty - \int_x^\infty \left(-e^{-t}\right)\left(-\frac{1}{t^2}\right) dt$$

$$= \frac{e^{-x}}{x} - \underbrace{\int_x^\infty \frac{e^{-t}}{t^2}\, dt}_{=:\, R_1(x)}$$

where $|R_1(x)| \leq \int_x^\infty \frac{e^{-t}}{x^2}\, dt$

$$= \frac{1}{x^2}\left(-e^{-t}\right)\Big|_x^\infty = \frac{e^{-x}}{x^2}$$

$$\Rightarrow \int_x^\infty \frac{e^{-t}}{t}\, dt = \frac{e^{-x}}{x} + o\left(\frac{e^{-x}}{x}\right) \quad \text{as } x \to \infty.$$

2. (a) Let A be an open subset and B be a closed subset of a metric space. Determine whether the following occur always, sometimes, or never; give proofs or (counter)-examples.

   (i) $A \setminus B \equiv \{x \in A : x \notin B\}$ is open.
   (ii) $A \setminus B$ is closed.
   (iii) $A \setminus B$ is both open and closed.
   (iv) $A \setminus B$ is neither open nor closed.

   (b) Let X, Y, and Z be metric spaces, and $f : Y \to Z$ and $g : X \to Y$ be continuous mappings. Use the topological characterization of continuity to show that the composition $f \circ g$ is continuous.

(5+5)

(a) (i) $A \setminus B = \underset{\text{open}}{A} \cap \underset{\text{open}}{B^c}$

So $A \setminus B$ is open as an intersection of a finite number of open sets.

(ii) Possible, e.g. $A = \mathbb{R}$, $B = \emptyset$ $\Rightarrow$ $A \setminus B = \mathbb{R}$

But not always, e.g. $A = \mathbb{R}$, $B = \{0\}$, so $A \setminus B$ not closed as 0 is a limit point of $A \setminus B$.

(iii) Possible, see (ii).

(iv) Impossible, see (i).

(b) Let $E \subset Z$ be open. f cont, so $U := f^{-1}(E) \subset Y$ open

g cont., so $V := g^{-1}(U) \subset X$ open

This implies that $(f \circ g)^{-1}(E)$ open $\Rightarrow$ $f \circ g$ cont.

$\square$

3. (a) Let $\{x_n\}$ be a converging sequence of elements in a metric space X with limit x. Show that the set

$$E = \{x\} \cup \{x_n : n \in \mathbb{N}\}$$

is compact.

(b) Prove that the boundary of a compact set is compact.

(5+5)

(a) Let $\{F_\alpha\}$ be an open cover of E.

Then $\exists\, F \in \{F_\alpha\}$ such that $x \in F$.

Since F open, $\exists\, \varepsilon > 0$ s.t. $N_\varepsilon(x) \subset F$.

Since $x_n \to x$, all but finitely many members of $\{x_n\}$ are contained in $N_\varepsilon(x)$, hence are covered by F. For the rest, it's obvious that a <u>finite</u> number of additional sets from $\{F_\alpha\}$ suffice to complete a cover for E.

$\square$

(b) Suppose E compact, hence E closed, hence $\partial E \subset E$

From homework: the boundary of any set is closed. (*)

Hence, ∂E is compact as a closed subset of a compact set (Theorem from class.)

Proof of (*), not required for full credit: Suppose not, i.e. $\exists\, x \in X$ limit point of ∂E, but $x \notin \partial E$. Then $\forall\, N_\varepsilon(x)$, $\varepsilon > 0$, $N_\varepsilon(x) \cap \partial E \neq \emptyset$. But then $N_\varepsilon(x)$ contains some $y \in \partial E$. Moreover, since $N_\varepsilon(x)$ open, it contains some $N_\delta(y) \subset N_\varepsilon(x)$ which, by definition of ∂E, contains points in E and E^c. So same for $N_\varepsilon(x)$. $\lightning$

Direct proof for part (b):

Let $E \subset X$ be compact.

Let $\{F_\alpha\}$ be an open cover of ∂E.

Then $\{F_\alpha\}$ together with $\text{int} E$ (which is open!) is an open cover of E. Since E compact, there exists an open subcover covering E, hence, covering ∂E as well.

$\Rightarrow \partial E$ is compact.

4. Let $f_n, g_n\colon \mathbb{R} \to \mathbb{R}$ be sequences of bounded continuous functions which converge uniformly to functions f and g, respectively.

   (a) Show that $f_n\, g_n \to f\, g$ uniformly as $n \to \infty$.

   (b) Give an example which shows that convergence may fail to be uniform when the assumption of boundedness is dropped.

(5+5)

(a) Let $\varepsilon > 0$.

$$|f_n g_n - fg| \leq |f_n g_n - f_n g| + |f_n g - fg|$$

Since $f_n \to f$ uniformly and f_n bounded, this bound may be uniform in n. ($|f_n| \leq |f_n - f_m| + |f_m|$. Now choose N large enough s.t. $|f_n - f_m| \leq 1$ $\forall n, m \geq N$. Then

$$\sup_{\substack{n \geq N,\\ x \in \mathbb{R}}} |f_n(x)| \leq \underbrace{1 + \sup_{x \in \mathbb{R}} |f_m|}_{=: M} \qquad \text{for } m \geq N \text{ fixed.}$$

WLOG, we can choose M possibly larger s.t. $\sup_{\substack{n \in \mathbb{N}\\ x \in \mathbb{R}}} |f_n(x)| \leq M$.)

Hence, $|f_n g_n - fg| \leq M\, |g_n - g| + \tilde{M}\, |f_n - f|$

(Choose $\tilde{M}$ as a bound for g by an argument just like the above.)

Now, by uniform convergence of f_n and g_n, $\exists\, N$ s.t. $|g_n(x) - g(x)| \leq \frac{\varepsilon}{2M}$ and $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2\tilde{M}}$ $\forall n \geq N,\ x \in \mathbb{R}$. Thus,

$$|f_n g_n - fg| \leq \varepsilon \quad \forall n \geq N,\ x \in \mathbb{R}$$

□

(b) $f_n = x + \frac{1}{n} = g_n$, so $f_n g_n = x^2 + 2\frac{x}{n} + \frac{1}{n^2} \to x^2 = fg$ ptwse, but not unif.

5. Consider the sequence of functions on $[0,1]$ defined by

$$f_n(x) = n \sin \frac{x}{n}.$$

(a) Show that there exists a uniformly converging subsequence of $\{f_n\}$.

(b) What is the limit function?

(5+5)

(a) $f_n'(x) = \cos \frac{x}{n}$

which is bounded uniformly $\forall x \in [0,1]$, $n \in \mathbb{N}$.

By the FTC, this implies equicontinuity.

($|f_n(y) - f_n(x)| \leq \int_x^y |f_n'(\tilde{x})| \, d\tilde{x} \leq |y-x|$,

so the δ in the ε-δ-definition of continuity can be chosen independent of n, this implies equicontinuity.)

Also, for $x \geq 0$, $\sin x \leq x$, so

$$|f_n(x)| \leq n \frac{x}{n} \leq x.$$

This implies pointwise boundedness of $\{f_n\}$.

The claim then follows by Arzelà-Ascoli.

(b) It suffices to compute the point-wise limit, i.e. x fixed:

$$\lim_{n\to\infty} n \sin\frac{x}{n} = \lim_{y\to 0} \frac{\sin(xy)}{y} = \lim_{y \to 0} \frac{x\cos(xy)}{1} = x$$

L'Hôpital's rule.

6. Let $f \in C([0,1])$. Show that there exists a sequence of polynomials $\{p_n\}$ satisfying $p_n(0) = f(0)$ such that $p_n \to f$ uniformly on $[0,1]$. (5)

By Weierstraß, $\exists$ a sequence of polynomials $\{q_n\}$ s.t. $q_n \to f$ uniformly on $[0,1]$. Set

$$p_n(x) = q_n(x) - q_n(0) + f(0)$$

$\Rightarrow$ $p_n(0) = f(0)$ by construction. Moreover,

$$|p_n(x) - f(x)| \leq \underbrace{|q_n(x) - f(x)|}_{\text{converging uniformly}} + \underbrace{|q_n(0) - f(0)|}_{\text{no } x\text{-dependence, so also converging uniformly}}$$

$\Rightarrow$ $p_n \to f$ uniformly.

7. Find a power series expansion centered at 0 for

$$\ln(1+x)$$

and determine the radius of convergence. (5)

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$$

$$\Rightarrow \int_0^x \frac{d}{dx}\ln(1+t)\,dt = \int_0^x \frac{1}{1+t}\,dt$$

The integrand on the RHS is the limit of the geometric series, which has radius of convergence 1, so can be integrated term-by-term for all $|x| < 1$.

$$\Rightarrow \ln(1+x) - \underbrace{\ln(1+0)}_{=0} = \int_0^x \left(1 - t + t^2 - t^3 + \dots\right) dt$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$$