Analysis HW 11 Solutions

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Problem 1

(a) Say $\lambda^T = (\lambda_1, ..., \lambda_n)$, $\nabla h(a) = (\partial_1 h(a), ..., \partial_d h(a))$ and $\nabla f(a) = (\partial_j f_i(a))_{i,j}$. So by Lagrange Multipliers Theorem, $\partial_i h(a) = \sum_{j=1}^{n} \lambda_j \partial_i f_j$.
So, $\frac{\partial H}{\partial x_i}(a, \lambda) = \partial_i h(a) - \lambda^T \partial_i f(a) = \partial_i h(a) - \sum_{j=1}^{n} \lambda_j \partial_i f_j = 0$. Similarly, $\frac{\partial H}{\partial \mu_i}(a, \lambda) = \lambda_i f_i(a) = 0$ as $f(a) = 0$. Thus (a, λ) is critical point of H.

(b) Put $h : \mathbb{R}^2 \to \mathbb{R}$ given by $h(x, y) = x^2 + y^2$ and $f(x, y) = x$. $H(x, y, \mu) = x^2 + y^2 - \mu x$. Then the point at which h assumes constrained extremum is $(0, 0)$ where $f'(0, 0) = (1, 0)^T$ which has maximal rank. And $\lambda = 0$ in this case, but

$$\text{Hess } H(0, 0, 0) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

which has two positive and one negative eigenvalues $(2, 1 + \sqrt{2} > 0, 1 - \sqrt{2} < 0)$, so is indefinite.

(*Note:* For dimension greater than 2, determinant alone is note enough to conclude if a point is extremum or not.

Problem 2

Going by the notation of problem 1, $h(x, y) = xy,\ f(x, y) = (x/a)^2 + (y/a)^2 - 1$ and $H(x, y, \lambda) = xy - \mu(x^2/a^2 + y^2/a^2 - 1)$. For simplicity I assume $a > 0, b > 0$. So for the constrained extremum, we set $\nabla H(x, y, \mu) = 0$ which gives three equations in three variables:$y = 2\mu x/a^2$, $x = 2\mu y/b^2$ and $x^2/a^2 + y^2/b^2 = 1$. Solving the equations gives

$$x = \pm a/\sqrt{2},\ y = \pm b/\sqrt{2},\ \text{and } \lambda = \pm ab/2.$$

This gives the maximum value as $ab/2$ when x and y have same sign and minimum value as $-ab/2$ when x and y have opposite sign. To see they are really the extrema, note that the constraint is an ellipse which is compact. So, the maxima and minima are assumed. And by the result of problem 1, the extrema have to be critical points of H so our conclusion should be correct.

Problem 3

(a) We perform the computation in two different order. Then apply Fubini's theorem. By partial fraction decomposition, setting

$$\frac{x}{(1 + xy)(1 + x^2)} = \frac{Ax + B}{1 + x^2} + \frac{C}{1 + xy},$$

where A, B and C are independent of x (but might depend on y), we get $A = \frac{1}{1+y^2},\ B = \frac{y}{1+y^2},\ C = \frac{-y}{1+y^2}$. Then integrating with respect to x (from 0 to 1) leaves us with (each term is a standard tabulated integral)

$$\int_0^1 \frac{1}{y^2 + 1} \left(\frac{\pi y}{4} - \log(y + 1) + \frac{\log 2}{2} \right) dy.$$

First and third terms can be integrated easily, which leaves us with

$$\frac{\pi}{8} \log 2 - I + \frac{\pi}{8} = \frac{\pi}{4} \log 2 - I, \tag{1}$$

where $I := \int\limits_0^1 \frac{\log(1+y)}{1+y^2}$. The standard indefinite integrals used in above computations are:

$$\int \frac{2xdx}{1+x^2} = \log 1 + x^2,$$

$$\int \frac{1}{1+x^2}dx = \arctan x$$

and

$$\int \frac{1}{1+ax}dx = \frac{\log 1 + ax}{a}.$$

Now its turn to compute the integral in different order: first y then x. Integrating first w.r.t y leaves us with:

$$\int_0^1 \frac{\log(1+x)}{1+x^2} = I. \tag{2}$$

Note that the original integrand is bounded and continuous (so integrable) in $[0,1]\times[0,1]$, so Fubini theorem applies, by which (1) = (2) which yields the required result indicated in the question.

(b) This is a straightforward application of integration by parts: note $D(\arctan x) = \frac{1}{1+x^2}$ and $D[\log(1+x)] = \frac{1}{1+x}$. So,

$$\int_0^1 \frac{\arctan x}{1+x}dx = \arctan x \log(1+x)|_{x=0}^1 - \int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{4}\log 2 - I.$$

Using the value of I evaluated from part a), we obtain the desired result.

Problem 4

(a)

$$f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}.$$

So using the substitution $x = y\tan\theta$, we get:

$$\int_0^1 f(x,y)dx = \int_0^{\tan^{-1}(1/y)} \left(\frac{1}{y} - \frac{2\cos^2\theta}{y}\right)d\theta.$$

Using the relation $2\cos^2\theta = 1+\cos(2\theta)$ and $\sin(2\theta) = 2\sin\theta\cos\theta$, the integral evaluates to:

$$-\sin\left(\tan^{-1}\left(\frac{1}{y}\right)\right)\cos\left(\tan^{-1}\left(\frac{1}{y}\right)\right) = -\frac{1}{1+y^2}.$$

(b) Since $1+y^2 \geq 1$ for all $y\in\mathbb{R}$, the integral we computed in a) extends continuously to all square $I=[0,1]^2$. And from a),

$$\int_0^1\int_0^1 f(x,y)dxdy = \int_0^1 -\frac{1}{1+y^2}dy = -\frac{\pi}{4}.$$

On the other hand,

$$\int_0^1\int_0^1 f(x,y)dydx = -\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dydx,$$

which, except for minus sign in front, is same as the integral we computed earlier with x and y swapped. So, it will be negative of what we obtained earlier, that is:

$$\int_0^1\int_0^1 f(x,y)dydx = \frac{\pi}{4}.$$

(c) No this does not contradict Fubini Theorem because $f(x,y)$ is not integrable. If we switch to polar coordinates, it immediately becomes apparent what the issue is: the integral becomes:

$$\int_I \frac{\cos(2\theta)}{r} dr d\theta.$$

Here θ varies from 0 to $\pi/2$. Instead, if we restrict θ between 0 and $\pi/4$, the integrand is always non-negative and the integral can be easily computed to be $+\infty$ (restrict r between 0 and 1, then the region we obtain will be subset of $I \cap \{\pi/4 \geq \theta \geq 0\}$). On the other hand, restricting θ between $\pi/4$ and $\pi/2$ yields integral as $-\infty$. Switching to polar coordinates is justified since the change of coordinates is a smooth diffeomorphism on the interior of our domain I.