Analysis II

Final Exam

May 31, 2019

1. Are the following statements true or false? If true, give a brief justification (in case the result is a named theorem or otherwise known from class or from the homework, you may simply state this). If false, present a counter-example.

   (a) An arbitrary union of closed sets is closed.

   (b) An arbitrary union of open sets is open.

   (c) A subset of a compact set is compact.

   (d) A subset of $\mathbb{R}^n$ is compact if it is bounded and closed.

   (e) A convex set is connected.

   (2 pts. each)

2. Are the following statements true or false? If true, give a brief justification (in case the result is a named theorem or otherwise known from class or from the homework, you may simply state this). If false, present a counter-example.

   (a) Let f_n be a uniformly convergent sequence of continuous functions on $I = [a, b]$. Then
   $$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x\,.$$

   (b) The statement from (a) with $I = [0, \infty)$.

   (c) Here and in the following, let $f\colon \mathbb{R}^n \to \mathbb{R}$. Suppose all partial derivatives of f exist at some point $x \in \mathbb{R}^n$. Then all directional derivatives exist at x.

   (d) Suppose all directional derivatives exist at $x \in \mathbb{R}^n$. Then f is differentiable at x.

   (e) Suppose f is twice continuously differentiable. Then the Hessian of f is symmetric.

   (2 pts. each)

3. Let X be the vector space of all bounded sequences $x = (x_1, x_2, \dots)$ endowed with the norm
   $$\|x\| = \sup_{i\in\mathbb{N}} |x_i|\,.$$

(a) Show that $\|\cdot\|$ is indeed a norm.

(b) Show that the set

$$B = \{x \in X \colon \|x\| \leq 1\}$$

is bounded and closed.

(c) Show that B is not compact.

(5+5+5)

4. Find the power series expansion centered at 0 for the function

$$f(x) = \frac{1}{x^2 + 4}$$

and determine its radius of convergence. (5)

5. Compute the derivative of the following maps.

(a) $f \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\boldsymbol{v}) = \boldsymbol{v}^T A \boldsymbol{v}$$

where A is a fixed $n \times n$ matrix, not necessarily symmetric.

(b) $f \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ defined by

$$f(A) = \boldsymbol{v}^T A \boldsymbol{v}$$

where $\boldsymbol{v} \in \mathbb{R}^n$ is fixed.

In each case, state the mapping properties (domain and range) of the derivative explicitly. (5+5)

6. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f cannot be injective.

Hint: Implicit function theorem. (10)

7. Maximize

$$f(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = xy + xz + yz = 1\,.$$

(10)

8. Let

$$D = \{(x, y) \colon x^2 + y^2 \leq 1\}$$

denote the unit disk in $\mathbb{R}^2$. Compute the integral

$$\int_D \cos(x^2 + y^2)\,\mathrm{d}x\,.$$

Hint: Polar coordinates. (5)

9. Compute the flux

$$\int_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S \,,$$

where $\boldsymbol{n}$ is the outward unit normal and

$$\boldsymbol{F} = \begin{pmatrix} z \, \cos x \, \sin y \\ -z \, \cos x \, \sin y \\ \frac{1}{2} \, z^2 \end{pmatrix}$$

through the surface of the unit ball in $\mathbb{R}^3$.

Hint: Divergence theorem.

(5)