1. Are the following statements true or false? If true, give a brief justification (in case the result is a named theorem or otherwise known from class or from the homework, you may simply state this). If false, present a counter-example.

   (a) An arbitrary union of closed sets is closed.

   (b) An arbitrary union of open sets is open.

   (c) A subset of a compact set is compact.

   (d) A subset of $\mathbb{R}^n$ is compact if it is bounded and closed.

   (e) A convex set is connected.

(2 pts. each)

(a) False. E.g. $I_n = [\frac{1}{n}, 1-\frac{1}{n}] \Rightarrow \bigcup_{n=1}^{\infty} I_n = (0,1)$ which is not closed; its closure is $[0,1]$.

(b) True. Theorem from class (or Rudin, Theorem 2.24 (a).)

(c) False, the subset may not be closed: $(0,1) \subset \underbrace{[0,1]}_{\text{compact}}$

(d) True: Heine-Borel theorem.

(e) True: A convex $\Rightarrow$ the line segment connecting any two points of A is contained in A $\Rightarrow$ A is path-connected $\Rightarrow$ A connected.

2. Are the following statements true or false? If true, give a brief justification (in case the result is a named theorem or otherwise known from class or from the homework, you may simply state this). If false, present a counter-example.

   (a) Let f_n be a uniformly convergent sequence of continuous functions on $I = [a, b]$. Then
   $$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x\,.$$

   (b) The statement from (a) with $I = [0, \infty)$.

   (c) Here and in the following, let $f\colon \mathbb{R}^n \to \mathbb{R}$. Suppose all partial derivatives of f exist at some point $x \in \mathbb{R}^n$. Then all directional derivatives exist at x.

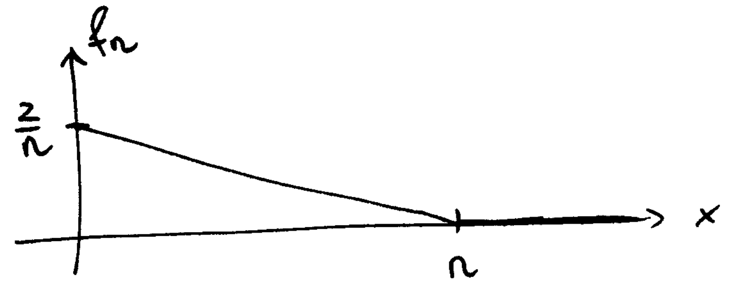
   (d) Suppose all directional derivatives exist at $x \in \mathbb{R}^n$. Then f is differentiable at x.

   (e) Suppose f is twice continuously differentiable. Then the Hessian of f is symmetric.

(2 pts. each)

(a) True, theorem from class (or Rudin, Theorem 7.16)

(b) False. Take, for example, the sequence f_n defined as



Then $\sup_{[0,\infty)} |f_n| = \frac{2}{n} \to 0$ as $n \to \infty$, so $f_n \to 0$ uniformly.

On the other hand, $\int_0^\infty f_n(x)\,dx = 1$ for all n.

$$\Rightarrow 1 = \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty \lim_{n\to\infty} f_n(x)\,dx = 0\,.$$

(c) False. E.g. $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x_1 = 0 \text{ or } x_2 = 0 \\ 1 & \text{otherwise} \end{cases}$$

At $x=0$, $\partial_1 f = \partial_2 f = 0$, but on any slanted line through the origin, f is not even continuous.

(d) False. E.g. $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x_1 > 0 \text{ and } x_2 > \sqrt{x_1} \\ 0 & \text{otherwise.} \end{cases}$$

Then on every line through the origin, there exists a neighborhood of 0 such that $f = 0$ in the intersection of the neighborhood with the line. So, any directional derivative at the origin exists and is zero.

Yet, f is not continuous, hence not differentiable, at 0.

(e) True, theorem from class (or Corollary to Rudin, Theorem 9.41)

3. Let X be the vector space of all bounded sequences $x = (x_1, x_2, \dots)$ endowed with the norm

$$\|x\| = \sup_{i \in \mathbb{N}} |x_i| .$$

(a) Show that $\|\cdot\|$ is indeed a norm.

(b) Show that the set

$$B = \{x \in X \colon \|x\| \leq 1\}$$

is bounded and closed.

(c) Show that B is not compact.

(5+5+5)

(a) (i) $\|x\| = 0 \Rightarrow |x_i| \leq 0 \Rightarrow |x_i| = 0 \Rightarrow x = 0$

(ii) $\|\lambda x\| = \sup_{i \in \mathbb{N}} |\lambda| |x_i| = |\lambda| \|x\|$

(iii) $\|x+y\| = \sup_{i \in \mathbb{N}} |x_i + y_i| \leq \sup_{i \in \mathbb{N}} (|x_i| + |y_i|) \leq \underbrace{\sup_{i \in \mathbb{N}} |x_i|}_{= \|x\|} + \underbrace{\sup_{i \in \mathbb{N}} |y_i|}_{= \|y\|}$

(b) B is bounded by construction. We prove that B is closed.

Let x be a limit point of B. Then $\forall \varepsilon > 0 \ \exists x_\varepsilon \in B$ s.t. $\|x - x_\varepsilon\| \leq \varepsilon$.

By the triangle inequality, $\|x\| - \|x_\varepsilon\| \leq \varepsilon \Rightarrow \|x\| \leq \varepsilon + \|x_n\| \leq 1 + \varepsilon$.

As ε is arbitrary, this implies $\|x\| \leq 1 \Rightarrow x \in B$.

(c) Let $x_n = (0, \dots, 0, \underset{\substack{\uparrow \\ n\text{-th position}}}{1}, 0, \dots) \in B$

Clearly $\|x_n - x_m\| = 1$ if $n \neq m$

$\Rightarrow \{x_n\}$ does not have a Cauchy subsequence, hence no converging subsequence. $\Rightarrow B$ not compact.

4. Find the power series expansion centered at 0 for the function

$$f(x) = \frac{1}{x^2 + 4}$$

and determine its radius of convergence. (5)

$$f(x) = \frac{1}{4} \frac{1}{1 - \left(-\frac{x^2}{4}\right)} = \frac{1}{4} \sum_{i=0}^{\infty} \left(-\frac{x^2}{4}\right)^i$$

This is the geometric series with base $-\frac{x^2}{4}$, so its radius of convergence is determined by

$$\frac{r^2}{4} = 1 \quad \Rightarrow \quad r = 2 .$$

5. Compute the derivative of the following maps.

(a) $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\boldsymbol{v}) = \boldsymbol{v}^T A \boldsymbol{v}$$

where A is a fixed $n \times n$ matrix, not necessarily symmetric.

(b) $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

$$f(A) = \boldsymbol{v}^T A \boldsymbol{v}$$

where $\boldsymbol{v} \in \mathbb{R}^n$ is fixed.

In each case, state the mapping properties (domain and range) of the derivative explicitly. (5+5)

(a) $\delta f(v) = \delta v^T A v + v^T A \delta v = v^T (A^T + A) \delta v$

In different notation:

$$f'(v) w = v^T (A + A^T) w$$

Here, $f' : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$

(b) $\delta f(A) = v^T \delta A \, v$ or $f'(A)B = v^T B v$

Here, $f' : \mathbb{R}^{n\times n} \to \mathcal{L}(\mathbb{R}^{n\times n}, \mathbb{R})$

Note that in this case, $f(A)$ is linear, so f' is a constant linear map (does not depend on A).

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f cannot be injective. (10)

   Hint: Implicit function theorem.

Suppose there exists $x \in \mathbb{R}^2$ s.t. $\partial_1 f(x) \neq 0$. Then, by the implicit function theorem, there exists a neighborhood of x_2 and a C^1 function g on this neighborhood s.t.

$$f(g(y), y) = \text{const}$$

$\Rightarrow$ f not injective.

We conclude that $\partial_1 f = 0$ everywhere, but then f is constant w.r.t. x_1, hence not injective either.

<u>Note:</u> The statement is actually true under the weaker assumption that f is only continuous. Here is a proof:

First, note that a continuous injective function on the real line is strictly monotonic. (Why?)

Now fix $a > 0$.

Let $I_1 = f((-a,a), 0)$ and

$I_2 = f(0, (-a,a))$

be the images of open intervals with the other coordinate fixed to zero.

Due to the remark above, I_1 and I_2 are open intervals.

Moreover, $f(0,0) \in I_1 \cap I_2$, so $I_1 \cap I_2$ is non-empty and open (as finite intersection of open sets), so it contains an open neighborhood of $f(0,0)$.

Every point in this neighborhood, except $f(0,0)$, has two pre-images: one in $(-a,a) \times \{0\}$, the other in $\{0\} \times (-a,a)$.

$\Rightarrow f$ not injective.

7. Maximize

$$f(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = xy + xz + yz = 1\,.$$

(10)

Necessary condition for extrema under a constraint comes from the Lagrange-multiplyer theorem: $\exists \lambda$ s.t.

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$$

$$\Rightarrow xyz = \lambda(y+z)x = \lambda(x+z)y = \lambda(x+y)z$$

$x=y$ or $z=0$ or $\lambda=0$

$y=z$ or $x=0$ or $\lambda=0$

The cases $x=0$, $z=0$ or $\lambda=0$ can be excluded as then $f=0$ which cannot be maximal. So we have $x=y=z$ and $3x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

For a maximum, we choose the +-sign, so $f_{max} = 3^{-\frac{3}{2}}$

Note that f, as a cont. function on the compact constraint set must have a maximum, i.e. the candidate maximum is a true one.

8. Let

$$D = \{(x,y)\colon x^2 + y^2 \leq 1\}$$

denote the unit disk in $\mathbb{R}^2$. Compute the integral

$$\int_D \cos(x^2+y^2)\,\mathrm{d}x\,.$$

Hint: Polar coordinates. (5)

$$\int_D \cos(x^2+y^2)\,dx = \int_0^1 \int_0^{2\pi} \cos(r^2)\, r\, d\theta\, dr$$

$$= 2\pi \int_0^1 \cos(r^2)\, r\, dr \qquad s = r^2 \Rightarrow ds = 2r\,dr$$

$$= \pi \int_0^1 \cos s \; ds$$

$$= \pi \sin(1)$$

9. Compute the flux

$$\int_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S \,,$$

where $\boldsymbol{n}$ is the outward unit normal and

$$\boldsymbol{F} = \begin{pmatrix} z \cos x \sin y \\ -z \cos x \sin y \\ \frac{1}{2} z^2 \end{pmatrix}$$

through the surface of the unit ball in $\mathbb{R}^3$.

Hint: Divergence theorem.

(5)

$$\operatorname{div} F = \nabla \cdot F = z(-\sin x) \sin y - z \cos x \cos y + z$$

$$= z(1 - \sin x \sin y - \cos x \cos y)$$

$$= z \cdot g(x,y)$$

$$\Rightarrow \int_{\partial D} F \cdot n \, dS = \int_D \operatorname{div} F \, dV = \int_D z \, g(x,y) \, dx \, dy \, dz$$

$$= \int_{x^2+y^2 \leq 1} \underbrace{\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z \, dz}_{=0 \text{ by symmetry}} \; g(x,y) \, dx \, dy$$

$$= 0$$