Analysis II

Homework 9

Due in class Tuesday, April 23, 2019

Recall the *inverse function theorem*. For $E \subset \mathbb{R}^n$ open and $f \in C^1(E, \mathbb{R}^n)$, suppose that $Df(a) \equiv f'(a)$ is invertible for some $a \in E$.[1] Then

(i) There exist open sets $U, V \subset \mathbb{R}^n$ with $a \in U$ such that $f\colon U \to V$ is bijective,

(ii) $f^{-1} \in C^1(V, U)$ and
$$Df^{-1}(y) = Df(x)^{-1} \tag{*}$$
with $y = f(x)$.

1. (From Rudin, Exercise 9.17.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f(x) = \begin{pmatrix} \mathrm{e}^{x_1} \, \cos x_2 \\ \mathrm{e}^{x_1} \, \sin x_2 \end{pmatrix} .$$
   (a) What is the range of f?
   (b) Show that the *Jacobian determinant*, $\det Df(x)$, is non-zero for every $x \in \mathbb{R}^2$. Thus every point in $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.
   (c) Put $a = (0, \pi/3)$ and $b = f(a)$. Find an explicit formula for f^{-1}, compute $Df(a)$ and $Df^{-1}(b)$, and verify formula (*) above.
   (d) What are the images under f of lines parallel to the coordinate axes?

2. (From Rudin, Exercise 9.18.) Answer analogous questions for the mapping defined by
$$f(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2\, x_1 \, x_2 \end{pmatrix} .$$

3. (From Rudin, Exercise 9.21.) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = 2\,x^3 - 3\,x^2 + 2\,y^3 + 3\,y^2 .$$

[1] Both notations, Df and f', are commonly used to denote the derivative of a function between normed vector spaces. Here, in the context of the inverse and the implicit function theorems, the D-notation has some advantages, where we understand that $Df^{-1}(x)$ is the derivative of the inverse map and $Df(x)^{-1}$ is the inverse of the derivative map at x.

(a) Find the four points in $\mathbb{R}^2$ at which the gradient $\nabla f = (\partial_x f, \partial_y f)$ is zero. Show that f has exactly one local maximum and one local minimum in $\mathbb{R}^2$.

(b) Let

$$S = \{(x,y) \in \mathbb{R}^2 \colon f(x,y) = 0\} .$$

Find these points of S which have no neighborhoods in which the equation $f(x,y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

4. (From the *inverse* to the *implicit function theorem*.) Let $X \subset \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be open and $f \in C^1(X \times Y, \mathbb{R}^n)$. Define $F \colon X \times Y \to \mathbb{R}^n \times \mathbb{R}^m$ as

$$F(x,y) = \begin{pmatrix} f(x,y) \\ y \end{pmatrix} .$$

(a) Show that DF has the block-matrix structure[2]

$$DF = \begin{pmatrix} D_x f & D_y f \\ 0 & I_m \end{pmatrix} ,$$

where I_m is the $m \times m$-identity matrix and $D_x f$ and $D_y f$ denote the generalized partial derivatives

$$D_x f = \begin{pmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_n} f_1 \\ \vdots & & \vdots \\ \partial_{x_1} f_n & \dots & \partial_{x_n} f_n \end{pmatrix} \quad \text{and} \quad D_y f = \begin{pmatrix} \partial_{y_1} f_1 & \dots & \partial_{y_m} f_1 \\ \vdots & & \vdots \\ \partial_{y_1} f_n & \dots & \partial_{y_m} f_n \end{pmatrix} .$$

(b) Conclude that DF is invertible if and only if $D_x f$ is invertible.

(c) Suppose that there exist $a \in X$ and $b \in Y$ such that $f(a,b) = 0$ and $D_x f(a,b)$ is invertible. Argue that there exists an open neighborhood V of $(0,b)$ and an open neighborhood U of (a,b) such that $F \colon U \to V$ is bijective with $F^{-1} \in C^1(V,U)$.

(d) Conclude that there exist an open neighborhood A of a, an open neighborhood B of b, and a function $g \colon B \to A$ such that

$$f(g(y), y) = 0$$

for every $y \in B$.

(e) Finally, argue that $g \in C^1(B,A)$ with

$$Dg(y) = -D_x f(g(y),y)^{-1} \, D_y f(g(y),y) .$$

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[2]Here, we identify the notion of derivative as a linear map with its matrix representation with respect to the standard basis. We make no attempt to use distinct notation for these two technically separate concepts because there is no useful general convention and, in practice, the precise meaning can be inferred from context.