Analysis II

Homework 7

Due in class Monday, April 1, 2019

1. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}^n$. Show that f is linear, i.e., that

$$f(\lambda x) = \lambda\, f(x)$$

for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

2. Let V be a finite-dimensional normed vector space. Recall from class the *general linear group*

$$\mathrm{GL}(V) = \{A \in L(V) \colon A \text{ is invertible}\}$$

Show that the map $\mathrm{inv}\colon \mathrm{GL}(V) \to L(V)$ defined by

$$\mathrm{inv}(A) = A^{-1}$$

is differentiable with

$$\mathrm{inv}'(A)B = -A^{-1}BA^{-1}$$

3. Let $E \subset \mathbb{R}^n$ be open and $f\colon E \to \mathbb{R}$ possesses partial derivatives $\partial_1 f, \dots, \partial_n f$ that are bounded on E. Show that f is continuous on E.

4. *Disconcerting Example 1.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

   (a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

   (b) Show that f is not differentiable at the origin.

5. *Disconcerting Example 2.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3\, y}{x^6 + y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

(b) Show that f is not continuous at the origin.

6. *Disconcerting Example 3.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2\, y}{x^4 + y^2}\, \sqrt{x^2 + y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{v}} f(0,0)$ for every $\boldsymbol{v} = (a,b) \in \mathbb{R}^2$. Is $\boldsymbol{v} \mapsto D_{\boldsymbol{v}} f(0,0)$ linear?

(b) Show that f is not differentiable at the origin.