Analysis II

Problem Set 6

Midterm Review Problems – Not for Credit

1. *Topic:* Integration by parts, asymptotics, Laplace's method (idea only)

   Show that, as $x \to \infty$,
   $$\int_x^\infty \mathrm{e}^{-x^4}\,\mathrm{d}x \sim \frac{\mathrm{e}^{-x^4}}{4\,x^3}\,.$$

2. *Topic*: Elementary point-set topology in metric spaces.

   (a) Let A be an open subset and B be a closed subset of a metric space. Determine whether the following occur always, sometimes, or never; give proofs or (counter)-examples.

   - $\overline{A \cup B}$ is open.
   - $\overline{A \cup B}$ is closed.
   - $\overline{A \cup B}$ is both open and closed.
   - $\overline{A \cup B}$ is neither open nor closed.

   (b) Let X be a metric and f be a continuous real-valued function on X. Prove that the zero-set of f, i.e., the set of all points $p \in X$ such that $f(p) = 0$, is closed.

3. *Topic*: Compactness

   (a) Let $\{E_\alpha\}$ be a family of compact subsets of a metric space X. Show that
   $$E = \bigcap_\alpha E_\alpha$$
   is compact.

   (b) Prove that every compact metric space has a countable dense subset.

4. *Topic*: Uniform convergence (elementary properties and examples)

   Let $f_n\colon [0,1] \to \mathbb{R}$ and $g_n\colon [0,1] \to [0,1]$ be sequences of continuous functions converging uniformly to $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to [0,1]$, respectively. Show that
   $$f_n \circ g_n \to f \circ g$$
   uniformly as $n \to \infty$.

5. *Topic*: Arzelà–Ascoli theorem

   Construct a proof of *Peano's existence theorem* in the following simplified version. Suppose $f \in C([-1,1])$. Then there exists $\delta > 0$ such that the differential equation

$$y'(t) = f(y(t))\,,$$
$$y(0) = 0$$

   has a solution, not necessarily unique, on the interval $[0,\delta]$.

   Proceed as follows. For every $n \in \mathbb{N}$, define the function $y_n(t)$ via the recursive process

$$y_n(t) = \begin{cases} 0 & \text{for } t \in [0, \delta/n]\,, \\ \int_0^t f(y_n(s - \delta/n))\,\mathrm{d}s & \text{for } t \in (\delta/n, \delta]\,. \end{cases}$$

   (a) Find $\delta > 0$ such that $\{y_n\}$ is well-defined.
   (b) Show that $\{y_n\}$ is point-wise bounded, cf. step (a), and equicontinuous.
   (c) Argue that there exists a subsequence converging uniformly to some $y \in C([0,\delta])$.
   (d) Show that y satisfies the integral equation

$$y(t) = \int_0^t f(y(s))\,\mathrm{d}s\,.$$

   (You may appeal to the result of Problem 4 above.)

   (e) Argue that this implies that y satisfies the differential equation in Peano's existence theorem.

6. *Topic*: Weierstraß polynomial approximation theorem

   Show that if f is continuous on $\mathbb{R}$, there exists a sequence of polynomials $\{p_n\}$ that converges to f uniformly on every bounded subset of $\mathbb{R}$.

7. *Topic*: Power series expansions and radius of convergence

   Find a power series expansion centered at 0 for

$$\frac{1}{(1+x^2)^2}$$

   and determine the radius of convergence.