Analysis II

Homework 5

Due in class Monday, March 11, 2019

1. Let
$$\delta_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n\,, \\ n/2 & \text{for } |x| \leq 1/n\,. \end{cases}$$
Show that δ_n is a δ-sequence, i.e., that
$$\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{d}x = 1$$
for every $n \in \mathbb{N}$, and
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\,\delta_n(x)\,\mathrm{d}x = f(0)$$
for every continuous function f defined on $\mathbb{R}$.

2. Let $\{f_n\}$ be a sequence of twice differentiable functions on $[0,1]$ such that $f_n(0) = f_n'(0) = 0$ for all $n \in \mathbb{N}$ and such that $|f_n''(x)| \leq 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$.

   Show that there exists a subsequence of $\{f_n\}$ which converges uniformly on $[0,1]$.

3. (Rudin, Exercise 7.20.) If f is continuous on $[0,1]$ and if
$$\int_0^1 f(x)\,x^n\,\mathrm{d}x = 0$$
for every $n = 0, 1, \ldots,$ prove that $f(x) = 0$ on $[0,1]$.

   Hint: The integral of the product of f with any polynomial is zero. Use the Weierstraß theorem to show that
$$\int_0^1 f^2(x)\,\mathrm{d}x = 0\,.$$

4. Find the radius of convergence for
$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}\,x^n\,.$$

5. Find an explicit expression for the power series

   (a) $\displaystyle\sum_{n=1}^{\infty} n^3\, x^n\,,$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}\,.$

   State the radius of convergence for each power series.

6. Find a power series representation centered at 2 for

$$\frac{1}{4\,x - x^2 - 3}\,.$$

   What is the radius of convergence?