Analysis II

Homework 4

Due in class Monday, March 4, 2019

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Consider the sequence of left translates $f_n(x) = f(x+n)$, where $n \geq 1$. Prove that f_n converges uniformly on $[0,\infty)$ to the constant function c if and only if $\lim_{x\to\infty} f(x) = c$.

2. Test the following series for uniform convergence on $\mathbb{R}$:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$

3. Let $f(x,y) = y^3\,\mathrm{e}^{-y^2 x}$ and define
$$\phi(y) = \int_0^\infty f(x,y)\,\mathrm{d}x\,.$$

   (a) Show that, at $y=0$,
$$\phi'(y) \neq \int_0^\infty \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,.$$

   (b) Which condition in the theorem from class on the interchange of differentiation and integration is violated?

4. Which of the following sequences of functions is equicontinuous on $\mathbb{R}$? Prove or disprove: (a) $f_n(x) = \sin(nx)$, (b) $f_n(x) = \sin(nx)/n$.

5. (Rudin, Exercise 7.16.) Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

6. (Rudin, Exercise 7.18.) Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on some interval $[a,b]$ and define, for $x \in [a,b]$,
$$F_n(x) = \int_a^x f_n(t)\,\mathrm{d}t\,.$$
Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a,b]$.