Analysis II

Homework 10

Due in class Tuesday, April 30, 2019

1. Recall the implicit function theorem: Let $X \subset \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be open and $f \in C^1(X \times Y, \mathbb{R}^n)$. Suppose there exist $a \in X$ and $b \in Y$ such that $f(a,b) = 0$ and $D_x f(a,b)$ is invertible. Then there exist an open neighborhood A of a, an open neighborhood B of b, and a function $g \in C^1(B, A)$ such that

$$f(g(y), y) = 0$$

for every $y \in B$, and

$$Dg(y) = -D_x f(g(y), y)^{-1} \, D_y f(g(y), y) \, .$$

   (a) Argue that if $f \in C^k(X \times Y, \mathbb{R}^n)$ for $k \geq 1$, then $g \in C^k(B, A)$.
   Hint: Refer to Cramer's rule for the inverse matrix.

   (b) Show that if r is a simple root of the polynomial

$$p(x) = a_0 + a_1 \, x + \cdots + a_n \, x^n \, ,$$

   then r is a smooth, i.e. C^∞ function of the coefficients $a_0, \ldots, a_n$.

2. Consider the equation

$$\sqrt{x^2 + y^2 + 2\, z^2} = \cos z$$

near $(0, 1, 0)$. Can you solve for x in terms of y and z? For z in terms of x and y?