Analysis II

Homework 1

Due in class Monday, February 11, 2019

1. Recall the mean value theorem of integral calculus: Suppose $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable and non-negative, and $f\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $\xi \in [a,b]$ such that
$$f(\xi) \int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)\,g(x)\,\mathrm{d}x\,.$$
Give an example each to show that the following assumptions cannot be generally dropped:
   (a) g is non-negative,
   (b) f is continuous.

2. Recall Taylor's theorem in the following form. Suppose $f \in C^{n+1}(I)$ for some open interval I. Then for all $a, x \in I$,
$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}}{j!}\,(x-a)^j + R_n(x)$$
where
$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n\, f^{(n+1)}(t)\,\mathrm{d}t\,.$$
   (a) Turn the derivation shown in class into a formal proof by induction.
   (b) Show that the remainder can also be written as
$$R_n(x) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-s)^n\, f^{(n+1)}(a + s(x-a))\,\mathrm{d}s\,.$$
   (c) Show that there exists $\xi \in [a,x]$ such that
$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\, f^{(n+1)}(\xi)\,.$$
   (This expression is known as the *Lagrange form* of the remainder.)

3. Show that the *gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1}\, \mathrm{e}^{-t}\, \mathrm{d}t$$

is defined for all $x > 0$ and satisfies $\Gamma(x+1) = x!$ for each $x \in \mathbb{N}$.

Hint: Use integration by parts to show that $\Gamma(x+1) = x\,\Gamma(x)$.

4. Verify the value of the *Gaussian integral*

$$\int_{-\infty}^\infty \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}\,.$$

Hint: There are many different proofs, most of which use multi-dimensional integration which we did (yet) cover. An elegant derivation which avoids all multi-dimensional integrals works along the following lines.

(a) Let

$$F(t) = \int_0^\infty \frac{\mathrm{e}^{-t^2(1+x^2)}}{1+x^2}\, \mathrm{d}x$$

and show that $F(0) = \frac{\pi}{2}$ and $\lim_{t\to\infty} F(t) = 0$.

(b) Let

$$J = \int_0^\infty \mathrm{e}^{-x^2}\, \mathrm{d}x$$

and show that $F'(t) = -2\,J\,\mathrm{e}^{-t^2}$. (Fully justify your manipulations!)

(c) Use the Fundamental Theorem of Calculus on the result from part (b).

5. Check the conditions for Laplace's method for the function

$$f(x) = x - \ln x - 1$$

which appears in the proof of Stirling's formula. I.e., check the following.

(a) f is strictly decreasing for $0 < x < 1$, strictly increasing for $x > 1$, and $f(1) = 0$.

(b) There are positive constants b and c such that $f(x) \geq b\,x$ for $x \geq c$.

(c) $f(x) = a\,(x-1)^2 + \psi(x)\,(x-1)^3$ where ψ is a bounded function for $x \in [1-\delta, 1+\delta]$ for some $\delta > 0$. Find a explicitly.