Applied Differential Equations and Modeling

Problem Set 8

Review – Not for Credit

1. Solve the initial value problem

$$t^3\, y' + 4\, t^2\, y = \mathrm{e}^{-t}\,,$$
$$y(-1) = 0\,.$$

   On which interval of time does the solution exist?

2. Consider the differential equation

$$y' = \frac{t}{y}\,.$$

   (a) Solve the equation with initial condition $y(0) = a$.

   (b) Determine how the interval of existence depends on the initial value a.

3. Consider a cylindrical water tank with cross sectional area A filled with water up to height h. There is a constant inflow into the tank at rate k. At the bottom, the tank has a hole with effective cross sectional area a. By Torricelli's principle, the leakage rate is $a\sqrt{2gh}$, where g is the constant of gravity.

   (a) Argue that the height h as a function of time satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{k - a\sqrt{2gh}}{A}\,.$$

   (b) Without solving the equation, determine how h behaves as $t \to \infty$. Does the tank empty out, reach a stable equilibrium, or overflow any finite height of the tank?

   (c) Find the general solution to the differential equation and use the formula to confirm your conclusion from part (b).

4. Find the general solution to the linear system of equations

$$x_1 - 2\,x_2 + 4\,x_3 = 2\,,$$
$$2\,x_1 - x_2 - 2\,x_3 = -1\,,$$
$$3\,x_1 - x_2 + 2\,x_3 = 1\,,$$
$$2\,x_1 + 6\,x_2 - 12\,x_3 = -6\,.$$

5. Consider the system of linear differential equations

$$\boldsymbol{x}' = \begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix} \boldsymbol{x}$$

(a) Write out the general solution,

(b) Find the solution with

$$\boldsymbol{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} .$$

6. Consider the system of nonlinear differential equations

$$x' = y \,,$$
$$y' = -x + \tfrac{1}{6}\, x^3 - y \,.$$

(a) Find all equilibrium points,

(b) for each equilibrium point, write out the linear system describing the evolution of small perturbations about the equilibrium points, compute its eigenvalues and, if real-valued, eigenvectors, and

(c) determine the stability of each equilibrium point and sketch the phase portrait to the extent possible.