Applied Differential Equations and Modeling

Homework 3

Due in class Tuesday, February 26, 2019

1. For each of the following equations, in the general form $y' = f(y)$, determine the equilibrium points where $f(y) = 0$ and classify each as stable ($f(y)$ changes sign from positive to negative at the equilibrium value for y) or unstable (f changes sign from negative to positive). Sketch a few solution curves (without trying to solve the equation) in the t-y plane.

   (a) $y' = y - y^2$

   (b) $y' = y\,(y-1)\,(y-2)$

   (c) $y' = \mathrm{e}^y - a$ with constant $a > 0$

2. For each of the following equations, determine the equilibrium points and classify each as stable, unstable, or semi-stable ($f(y)$ does not change sign at the equilibrium value for y). Sketch a few solution curves (without trying to solve the equation) in the t-y plane.

   (a) $y' = y^2\,(y^2 - a^2)$ with constant $a \neq 0$

   (b) $y' = y^2\,(y-a)^2$ with constant $a \neq 0$

   Discussion point (extra credit): When discussing semi-stable equilibrium points, how can you be sure that the solution, coming from the "stable side", will converge toward the equilibrium point as time goes to infinity rather than "moving through it"? As you know from Calculus, there are functions, e.g. $y(t) = t^3$, where a zero of the first derivative does not correspond to an extreme value; the function is strictly increasing everywhere even though the first derivative is zero at a point. Can the same happen here?

3. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form
$$y' + p(t)\,y = q(t)\,y^n$$
   and is called Bernoulli's equation.

(a) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

(b) Use this method to solve the initial value problem

$$t^2\, y' + 2t\, y - y^3 = 0\,, \quad y(1) = 1\,.$$

(c) Use this method to find, by a different method than the one explained in class, the solution to the logistic differential equation

$$y' = r\, y \left(1 - \frac{y}{K}\right), \quad y(0) = y_0\,.$$

4. Rewrite the following differential equations as a system of first order differential equations of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{x}, t)\,,$$

where $\boldsymbol{x}$ and $\boldsymbol{F}$ are vector-valued functions of the form

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{F}(\boldsymbol{x}, t) = \begin{pmatrix} F_1(x_1, \ldots, x_n, t) \\ \vdots \\ F_n(x_1, \ldots, x_n, t) \end{pmatrix}.$$

(a) $y''' + t\, y'' + \sin t\, y' + y = 0$

(b) The second-order system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(v\, \frac{\mathrm{d}v}{\mathrm{d}t}\right) = 0\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\, \frac{\mathrm{d}v}{\mathrm{d}t}\right) = u\,.$$

For each system: Is it autonomous? Is it linear?