Applied Differential Equations and Modeling

Homework 10

Due in class Tuesday, April 30, 2019

1. Find the Laplace transform of the given function.

   (a) $f(t) = t^{10}$

   (b) $f(t) = \mathrm{e}^{2t} \cos 3t$

   (c) $f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t > 1 \end{cases}$

   (d) $f(t) = t^n \, \mathrm{e}^{at}$

2. Show that the Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L} \int_0^t f(\tau) \, \mathrm{d}\tau = \frac{1}{s} \, \mathcal{L}(f)$$

   assuming that the transforms on the left and on the right hand sides are well defined.

3. Apply the Laplace transform to the given initial value problem, and solve the resulting algebraic expression for the Laplace transform of the solution. (Note: the inverse transform, which is necessary to find the solution itself, will be discussed next week, it is not required here.)

   (a) $y'' - y' - 6y = 0$
   with $y(0) = 1$, $y'(0) = -1$

   (b) $y'' + \omega^2 \, y = \cos 2t$
   for $\omega^2 \neq 4$ with $y(0) = 1$, $y'(0) = 0$

   (c) $y'''' - 4y = 0$
   with $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 0$

4. For a function $f(t)$, write $F(s)$ to denote the Laplace transform. Prove the following.

   (a) $\mathcal{L}\left(\dfrac{f(t)}{t}\right) = \displaystyle\int_s^\infty F(u) \, \mathrm{d}u$

   (b) $\mathcal{L}(f(ct)) = \dfrac{1}{c} \, F\left(\dfrac{s}{c}\right)$