1. Solve the initial value problem

$$t\,y' + (t+1)\,y = 2\,t\,e^{-t},$$
$$y(1) = 1\,.$$

On which interval of time does the solution exist? (10)

Standard form:

$$y' + \left(1+\frac{1}{t}\right)y = 2e^{-t}$$

Integrating factor:

$$\mu = e^{\int (1+\frac{1}{t})\,dt} = e^{t+\ln t} \qquad (t>0\,!)$$

$$= te^{t}$$

$$\Rightarrow \frac{d}{dt}(te^{t}y) = 2(te^{t})e^{-t}$$
$$= 2t$$

$$\Rightarrow \int_1^t \frac{d}{dt}(te^{t}y)\,dt = \int_1^t 2t\,dt$$

$$\Rightarrow te^{t}y(t) - e\underbrace{y(0)}_{=1} = t^2-1$$

$$\Rightarrow y(t) = (t^2+e-1)\frac{e^{-t}}{t} = e^{-t}\left(t+\frac{e-1}{t}\right)$$

The solution has a vertical asymptote at $t=0$, initial condition is given at $t_0=1$, so interval of existence is $(0,\infty)$.

2. Consider the differential equation

$$y' + y^3 = 0\,.$$

(a) Solve the equation with initial condition $y(0) = a$.

(b) Determine how the interval of existence depends on the initial value a.

(10)

(a) $\frac{dy}{y^3} = -dt$

$\Rightarrow \int_a^{y(t)} \frac{dy}{y^3} = -t$

$\Rightarrow -\frac{1}{2} y^{-2} \Big|_a^{y(t)} = -t$

$\Rightarrow \frac{1}{y^2(t)} = \frac{1}{a^2} + 2t$ for $a \neq 0$!

$\Rightarrow y(t) = \frac{1}{\pm\sqrt{\frac{1}{a^2} + 2t}}$ (*)

(b) $a > 0$: Choose $+$-sign in (*), solution exists for all $\frac{1}{a^2} + 2t > 0 \Rightarrow t > -\frac{1}{2a^2}$

$a < 0$: Choose $-$-sign in (*), interval of existence as above.

$a = 0$: Solution is $y = 0$ (which is unique), interval of existence is not restricted.

3. Consider the *Gompertz growth model*

$$\frac{dy}{dt} = r\,y\,\ln\frac{K}{y},$$

where r and K are positive constants.

(a) Find the equilibrium points and classify each as stable or unstable. Sketch a few solution curves in the t-y plane.

(b) Find the general solution to the differential equation and use the formula to confirm your conclusion from part (a).

(c) Give an interpretation of the constant K when the model is used to describe population growth.

(5+5+5)

(a) $r y \ln\frac{K}{y} = 0 \quad \Rightarrow \quad y=0 \quad$ or $\quad y=K$

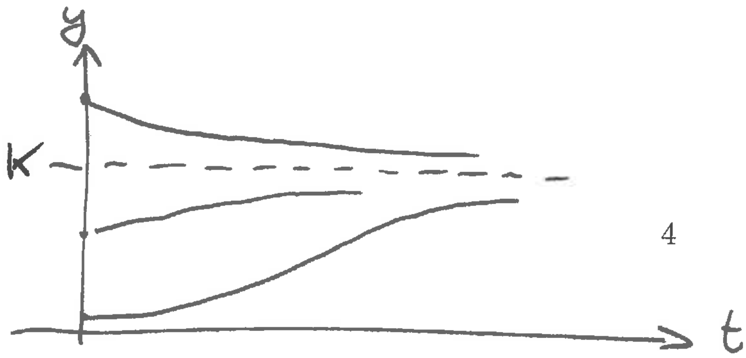
<u>Note:</u> The equilibrium point $y=0$ exists in the sense that

$$\lim_{y\to 0^+} y \ln\frac{K}{y} = 0.$$

For $0<y<K$, $\quad y\ln\frac{K}{y} > 0$,

for $y>K$, $\quad y\ln\frac{K}{y} < 0$

Thus, $y=0$ is unstable, $y=K$ is stable



(b) $$\int_{y(0)}^{y(t)} \frac{dy}{y \ln \frac{K}{y}} = \int_0^t r\,dt \qquad\qquad u = \ln\frac{K}{y}$$

$$\Rightarrow du = -\frac{1}{y}\,dy$$

$$\Rightarrow -\int_{\ln \frac{K}{y(0)}}^{\ln\frac{K}{y(t)}} \frac{d\cancel{y}u}{\cancel{y}u} = rt$$

$$\Rightarrow \ln|u| \Big|_{\ln\frac{K}{y(0)}}^{\ln\frac{K}{y(t)}} = -rt$$

$$\Rightarrow \ln \frac{\ln\frac{K}{y(t)}}{\ln\frac{K}{y(0)}} = -rt$$

(Note: From (a), $\ln\frac{K}{y(t)}$ and $\ln\frac{K}{y(0)}$ have the same sign, so that their ratio is always positive!)

$$\Rightarrow \ln\frac{K}{y(t)} = \ln\frac{K}{y(0)}\, e^{-rt}$$

$$\Rightarrow y(t) = K \underbrace{e^{\ln\frac{y(0)}{K}\underbrace{e^{-rt}}_{\to 0 \text{ as } t\to\infty}}}_{\to 1 \text{ as } t\to\infty}$$

$$\Rightarrow \lim_{t\to\infty} y(t) = K \qquad (y(0) > 0)$$

(c) The qualitative behavior is like for the logistic differential equation, so K could be interpreted as the capacity of an eco-system.

4. Find the general solution to the linear system of equations

$$\begin{aligned} x_1 + x_2 + 2\,x_3 &= 3\,, \\ x_1 + 2\,x_2 + x_3 - x_4 &= 4\,, \\ -x_1 - 3\,x_3 - x_4 &= -2\,. \end{aligned}$$

(10)

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & -1 & 4 \\ -1 & 0 & -3 & -1 & -2 \end{array}\right)$$

$$\xrightarrow[\substack{-R1+R2 \to R2 \\ R1+R3 \to R3}]{} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{-R2+R3 \to R3 \\ R1-R2 \to R1}]{} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So general solution reads

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} , \quad \lambda, \mu \in R$$

5. Consider the system of linear differential equations

$$x' = \underbrace{\begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix}}_{=:A} x$$

(a) Write out the general solution,

(b) Find the solution with

$$x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(5+5)

(a) Eigenvalues:

$$\det(A - \lambda I) = (-2-\lambda)(1-\lambda) + 4 = -2 - \lambda + 2\lambda + \lambda^2 + 4$$

$$= \lambda^2 + \lambda + 2$$

$$\Rightarrow \lambda_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$$

Eigenvectors:

For λ_+: $\begin{pmatrix} -2 + \frac{1}{2} - i\frac{\sqrt{7}}{2} & -2 \\ 2 & 1 + \frac{1}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{3}{2} - i\frac{\sqrt{7}}{2} & -2 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow V_+ = \begin{pmatrix} 2 \\ -\frac{3}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix}$$

For λ_-: $V_- = \begin{pmatrix} 2 \\ -\frac{3}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix}$ (must be complex-conjugate of V_+)

$$\Rightarrow x(t) = c_1 V_+ e^{\lambda_+ t} + c_2 V_- e^{\lambda_- t}$$

(b) $$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -\frac{3}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -\frac{3}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ -\frac{3}{2}-i\frac{\sqrt{7}}{2} & -\frac{3}{2}+i\frac{\sqrt{7}}{2} & 1 \end{array}\right) \xrightarrow[\left(\frac{3}{4}+i\frac{\sqrt{7}}{4}\right)R1+R2\to R2]{\frac{R1}{2}\to R1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & i\sqrt{7} & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -i\frac{1}{\sqrt{7}} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{i}{\sqrt{7}} \\ 0 & 1 & -\frac{i}{\sqrt{7}} \end{array}\right)$$

$$\Rightarrow \quad c_1 = \frac{i}{\sqrt{7}}, \quad c_2 = -\frac{i}{\sqrt{7}}$$

6. Consider the system of nonlinear differential equations

$$x' = 5x - xy,$$
$$y' = xy - y.$$

(a) Find all equilibrium points,

(b) for each equilibrium point, write out the linear system describing the evolution of small perturbations about the equilibrium points, compute its eigenvalues and, if real-valued, eigenvectors, and

(c) determine the stability of each equilibrium point and sketch the phase portrait to the extent possible.

(5+5+5)

(a) $5x - xy = 0 \Rightarrow x(5-y) = 0 \Rightarrow x = 0$ or $y = 5$

$xy - y = 0 \Rightarrow y(x-1) = 0 \Rightarrow y = 0$ or $x = 1$

$\Rightarrow$ equilibrium points are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

(b) For $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, linear system has matrix $A = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

By direct inspection, eigenvalues are $\lambda_1 = 5$ $\lambda_2 = -1$
with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, write $x = 1 + \xi$, $y = 5 + \eta$

$$\Rightarrow \xi' = 5(1+\xi) - (1+\xi)(5+\eta) = 5 + 5\xi - 5 - 5\xi - \eta - \xi\eta$$
$$= -\eta - \xi\eta$$

$$\eta' = (1+\xi)(5+\eta) - 5 - \eta = 5 + 5\xi + \eta + \xi\eta - 5 - \eta = 5\xi + \xi\eta$$

So linear system has matrix $A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = \lambda^2 + 5$

$\Rightarrow \lambda_{\pm} = \pm i\sqrt{5}$

$\Rightarrow$ This equilibrium point is a center.

(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable (has positive real eigenvalue)

$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ is stable (it's a center)