1. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} .$$

(10)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R1+R2 \to R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R2+R3 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

2. Solve the initial value problem

$$y' - y \sin t = \sin t \, ,$$
$$y(0) = \mathrm{e}^{-1} \, .$$

(10)

Integrating factor: $\mu = e^{-\int \sin t \, dt} = e^{\cos t}$

$$\Rightarrow \frac{d}{dt}\left(y e^{\cos t}\right) = \sin t \, e^{\cos t}$$

$$\Rightarrow y(t) e^{\cos t} - y(0) e^{1} = \int_0^t \sin \tilde{t} \, e^{\cos \tilde{t}} \, d\tilde{t}$$

$u = \cos \tilde{t}$
$du = -\sin \tilde{t} \, d\tilde{t}$

$$= -\int_1^{\cos t} e^{u} \, du$$

$$= e^{1} - e^{\cos t}$$

$$\Rightarrow y(t) e^{\cos t} = 1 + e - e^{\cos t}$$

$$\Rightarrow y(t) = (1+e) e^{-\cos t} - 1$$

3. Consider the differential equation

$$y' = \mathrm{e}^y \,.$$

(a) Solve the equation with initial condition $y(0) = a$.

(b) Determine how the interval of existence depends on the initial value a.

(10)

$$\frac{dy}{e^y} = dt$$

$$\Rightarrow \int_{y(0)}^{y(t)} e^{-y}\, dy = \int_0^t dt$$

$$\Rightarrow -e^{-y(t)} + e^{-y(0)} = t$$

$$\Rightarrow e^{-y(t)} = e^{-y(0)} - t$$

$$\Rightarrow y(t) = -\ln(e^{-a} - t)$$

We require that $e^{-a} - t > 0$

$$\Rightarrow t < e^{-a}$$

The solution exists for all $t < e^{-a}$.

4. Consider the system of linear differential equations

$$\boldsymbol{x}' = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}}_{=: A} \boldsymbol{x} .$$

(a) Write out the general solution.

(b) Find the solution with

$$\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

(5+5)

(a) eigenvalues: $\det(A - \lambda I) = (2-\lambda)(-\lambda) + 1$

$$= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$\Rightarrow \lambda = 1$

eigenvector: $A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

$\Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by inspection

generalized eigenvector: $(A - \lambda I) w = v$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow w = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

general solution takes the form $x(t) = c_1 v e^{\lambda t} + c_2 (w + tv) e^{\lambda t}$

(b) $x(0) = c_1 v + c_2 w$

Thus, we require

$$\begin{pmatrix} \vdots & \vdots \\ v & w \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right)$$

$\Rightarrow c_1 = 0, \quad c_2 = -1$

$$\Rightarrow x(t) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} t \right) e^t$$

5. Consider the forced harmonic oscillator described by the differential equation

$$y'' + y = g\,.$$

For $t < 0$, let $g(t) = 0$, so that $y(0) = 0$ and $y'(0) = 0$. At $t = 0$, the forcing is instantaneously increased to $g(t) = \frac{1}{2}$ and maintained at this level until, at the later time $t = c$, the forcing is instantaneously increased to its final value $g(t) = 1$.

(a) Use the Laplace transform to find the solution $y(t)$ assuming $c > 0$.

(b) How should you choose c so that the system is in a steady state (i.e., does not oscillate) for all $t > c$?

(5+5)

(a) $g(t) = \frac{1}{2} u(t) + \frac{1}{2} u(t-c)$

$\Rightarrow \mathcal{L}\{g\} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s} e^{-cs}$

Take Laplace transform of equation:

$$s^2 Y + Y = \frac{1}{2}\frac{1}{s}(1 + e^{-cs})$$

$$\Rightarrow Y(s) = \frac{1}{2} \underbrace{\frac{1}{s}\frac{1}{1+s^2}}_{= \frac{1}{s} - \frac{s}{1+s^2}} (1+e^{-cs})$$

$$\Rightarrow y(t) = \frac{1}{2}u(t) - \frac{1}{2}\cos t\, u(t) + \frac{1}{2}u(t-c) - \frac{1}{2}\cos(t-c)\, u(t-c)$$

$$= g(t) - \frac{1}{2}\left(\cos t\, u(t) + \cos(t-c)\, u(t-c)\right)$$

(b) For $t > c$, we need $\cos t + \cos(t-c) = 0$

So $c = \pi$ or, more generally, $c = \pi + 2\pi n$ for integer $n \geq 0$.

6. (a) Compute, without consulting the table of Laplace transforms, the Laplace transforms of the unit step function $u(t-c)$ and the delta-function $\delta(t-c)$ assuming $c > 0$.

(b) In which sense can you say that $u' = \delta$?

(5+5)

(a) $$\mathcal{L}\{u_c\} = \int_0^\infty e^{-st}\, u(t-c)\, dt = \int_c^\infty e^{-st} = -\frac{1}{s} e^{-st} \Big|_c^\infty$$

$$= \frac{1}{s} e^{-cs}$$

$$\mathcal{L}\{\delta_c\} = \int_0^\infty e^{-st}\, \delta(t-c)\, dt = e^{-cs}$$

In general, $\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0)$ (*)

Here, $\mathcal{L}\{\delta_c\} = s\,\mathcal{L}\{u_c\} - \underbrace{u_c(0)}_{=0}$ (**)

(b) Comparing (*) and (**), we see that $u_c' = \delta_c$ in the sense that their Laplace transforms coincide.

Note, however, that $u_c(t)$ is not differentiable (not even continuous) in the usual sense at $t = c$.

7. Consider a differential equation of the form

$$y'' + a\,y' + b\,y = g\,.$$

Show that the initial condition $y'(0) = 1$ can be replaced by adding the unit impulse function $\delta(t)$ to the forcing function g and setting $y'(0) = 0$. (5)

Take the Laplace transform:

$$s^2 Y - s y(0) - \underbrace{y'(0)}_{=1} + a(sY - y(0)) + bY = G$$

$$\Rightarrow \quad s^2 Y - s y(0) + a(sY - y(0)) + bY = G + 1$$

Now take inverse Laplace transform assuming $y'(0) = 0$

$$\Rightarrow \quad y'' + ay' + by = g + \delta$$

(Cf. Problem 6 for the Laplace transform of δ.)

8. Consider the system of nonlinear differential equations

$$x' = -y\,(1-x)\,,$$
$$y' = x - y^2\,.$$

(a) Find all equilibrium points,

(b) for each equilibrium point, write out the linear system describing the evolution of small perturbations about the equilibrium points, compute its eigenvalues and, if real-valued, eigenvectors, and

(c) determine the stability of each equilibrium point and sketch the phase portrait to the extent possible.

(5+5+5)

(a) $-y\,(1-x) = 0 \quad \Rightarrow \quad y = 0$ or $x = 1$

$x - y^2 = 0$

$\Rightarrow$ equilibrium points are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by inspection

at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: write $\begin{array}{l} x = 1+\xi \\ y = 1+\eta \end{array} \Rightarrow \begin{array}{l} \xi' = -(1+\eta)(1-1-\xi) = \xi + \dots \\ \eta' = 1+\xi-(1+\eta)^2 = \xi - 2\eta + \dots \end{array}$

$\Rightarrow$ matrix is $\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$

at $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$: write $\begin{array}{l} x = 1+\xi \\ y = -1+\eta \end{array}$ 10 $\Rightarrow \begin{array}{l} \xi' = -(-1+\eta)(1-1-\xi) = -\xi + \dots \\ \eta' = 1+\xi-(\eta-1)^2 = \xi + 2\eta + \dots \end{array}$

$\Rightarrow$ matrix is $\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$

(c) • at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: characteristic polynomial is $p(\lambda) = \lambda^2 + 1$,

so the roots are $\lambda = \pm i$.

$\Rightarrow$ it is a center, thus stable

• at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: eigenvalues are $\lambda = 1, -2$ (by inspection - diagonal entries of a triangular matrix)

For $\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

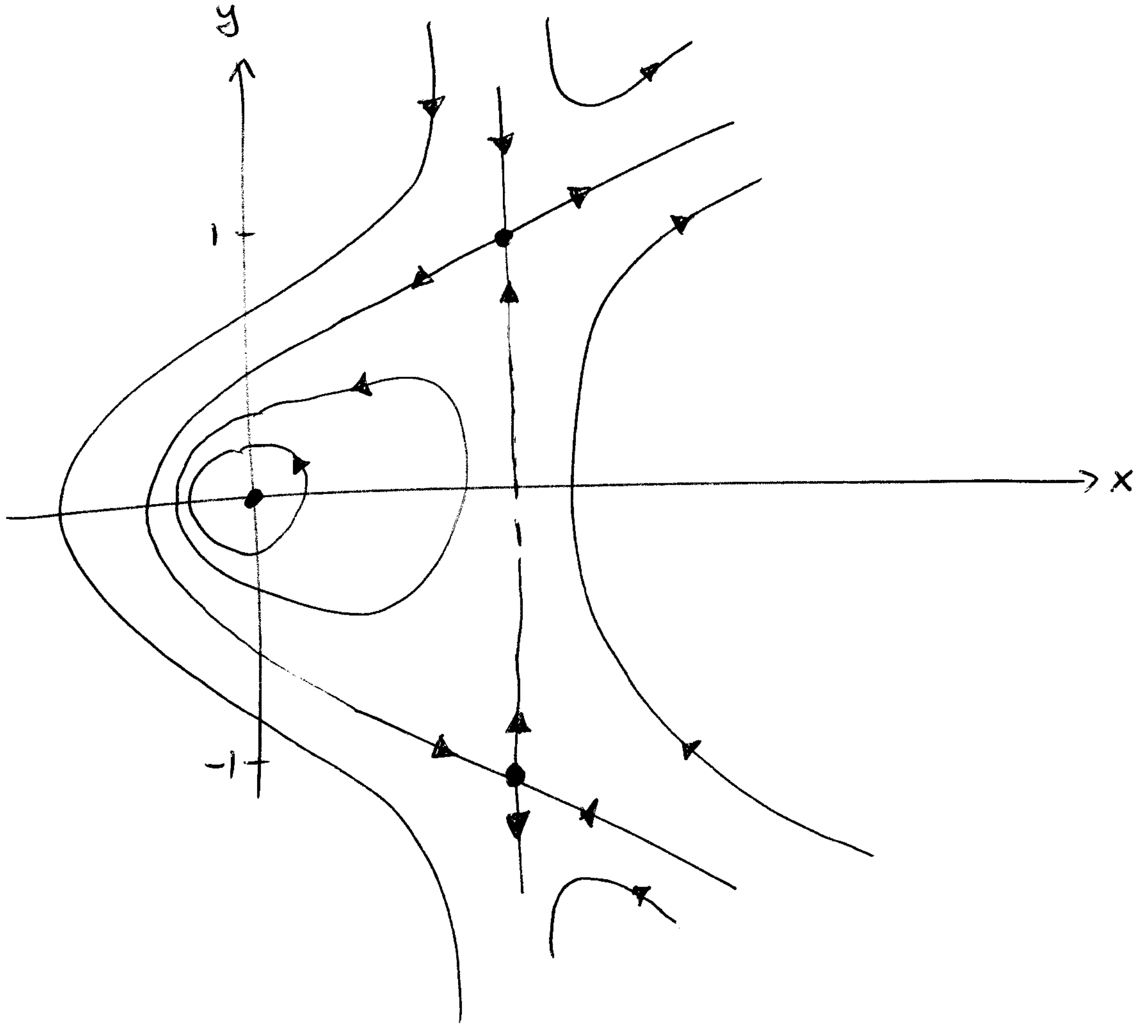
for $\lambda = -2$: $A - \lambda I = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

• at $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$: eigenvalues are $\lambda = -1, 2$

For $\lambda = -1$: $A - \lambda I = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

for $\lambda = 2$: $A - \lambda I = \begin{pmatrix} -3 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So the two other equilibrium points are saddle points.



phase portrait