Differential Equations

Homework 8

Due in class Tuesday, May 15, 2018

1. (Verhulst, Exercise 13.7.) Consider the so-called Brusselator system,

$$\dot{x} = a - (b+1)\,x + x^2\,y\,,$$
$$\dot{y} = b\,x - x^2\,y$$

in which x and y are concentrations (so $x, y \geq 0$), a and b are positive parameters. Does the Brusselator admit the possibility of an Andronov–Hopf bifurcation?

2. (Verhulst, Exercise 13.9.) We are looking for bifurcation phenomena in the system

$$\dot{x} = (1+a^2)\,x + (2-6a)\,y + f(x,y)\,,$$
$$\dot{y} = -x - 2\,y + g(x,y)$$

with parameter a; f and g can be developed into Taylor series near $(0,0)$ starting with quadratic terms.

   (a) For which values of a does the Poincaré–Lyapunov theorem guarantee asymptotic stability of $(0,0)$?

   (b) For which values of a is it possible to have a bifurcation of $(0,0)$?

   (c) For which values of a does a center manifold exist?

   (d) For which values of a is it possible to have an Andronov–Hopf bifurcation?

3. Fix $\sigma > 0$. Suppose that $f(z)$ is a 1-periodic function, analytic on every strip

$$S_\rho = \{z \in \mathbb{C} \colon |\mathrm{Im}\, z| < \rho\}$$

for some $\rho > \sigma$. Then the Fourier coefficients

$$f_n = \int_0^1 \mathrm{e}^{2\pi i n x}\, f(x)\, \mathrm{d}x$$

satisfy

$$|f_n| \leq \mathrm{e}^{-2\pi\sigma|n|}\, \|f\|_\sigma$$

where

$$\|f\|_\sigma = \sup_{z \in S_\sigma} |f(z)| \,.$$

Hint: Write $f(z) = g(\mathrm{e}^{2\pi i z})$ and set $w = \mathrm{e}^{2\pi i z}$. Argue that

$$f_n = \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{g(w)}{w^{n+1}} \, \mathrm{d}w \,.$$

Then use Cauchy's theorem to deform the circle of integration to its maximal extent.

4. Prove the following statement. Suppose h is 1-periodic and analytic on S_σ for some $\sigma > 0$. Set $H(x) = x + h(x)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|h\| < \delta$, then H is invertible, H^{-1} is analytic on $S_{\sigma-\varepsilon}$, and it takes the form

   $$H^{-1}(x) = x - h(x) + g(x)$$

   where

   $$\|g\|_{\sigma-\varepsilon} \leq \frac{c}{\varepsilon} \, \|h\|_\sigma^2$$

   for some constant c.

   (For notation, please refer to Problem 3.)

   Hint: Show that $g(x+h(x)) = h(x+h(x)) - h(x)$, then use the Fundamental Theorem of Calculus and finally a Cauchy estimate for h'.