Differential Equations

Homework 7

Due in class Thursday, May 3, 2018

1. Fill in the details of Verhulst, Example 13.9: Consider the augmented system

$$\dot{x} = \mu\, x - x^3 + x\, y\,,$$
$$\dot{y} = -y + y^2 - x^2\,,$$
$$\dot{\mu} = 0\,.$$

Near the origin, the center manifold can be parameterized by

$$y = h(x, \mu)\,.$$

(a) Writing h as a power series in x and μ, show that

$$\dot{u} = -x^2 + \text{higher order terms}\,.$$

(b) Show that the dynamics on the center manifold is given by

$$\dot{x} = \mu\, x - 2\, x^3 + O(x^3)\,,$$
$$\dot{\mu} = 0\,.$$

(c) Determine the critical points of the reduced system and their stability as a function of μ.

2. (Verhulst, Exercise 13.3.) Find an approximation to the center manifold W_{c} near the origin for the system

$$\dot{x} = -y + x\, z - x^4\,,$$
$$\dot{y} = x + y\, z + x\, y\, z\,,$$
$$\dot{z} = -z - (x^2 + y^2) + z^2 + \sin x^3\,.$$

Use the Center Manifold Theorem to determine whether the origin is a stable equilibrium point.

3. Consider the second order equation

$$y'' + y' - \mu\, y + y^2 = 0$$

with parameter μ.

(a) Write the equation as a first order system.

(b) Find the eigenvalues and eigenvectors of the system linearized about the origin.

(c) Explicitly transform the system into new coordinates u and v such that the linear part is diagonal.

Hint: You should find (with appropriate definitions of u and v), that

$$\dot{u} = \mu\,(u+v) - (u+v)^2\,,$$
$$\dot{v} = -v - \mu\,(u+v) + (u+v)^2\,.$$

(d) As in Problem 1 above, augment the system by the parameter equation $\dot{\mu}$ and approximate the center manifold in the u-v-μ phase space near the origin.

(e) Show that the local dynamics on the center manifold is given by

$$\dot{u} = \mu\,u - u^2 + O(u^3)\,.$$

(f) Find the critical points of the reduced dynamics and determine their stability; draw a bifurcation diagram in the μ-u plane.