Differential Equations

Homework 6

Due in class Tuesday, April 17, 2018

1. (Verhulst, Exercise 10.2.) Consider the mathematical pendulum

$$\ddot{x} + \sin x = 0$$

with initial conditions $x(0) = a$, $\dot{x}(0) = 0$.

(a) Setting $x = a\,y$, show that the system is equivalent to

$$\ddot{y} + y = \frac{a^2}{6}\,y^3 + O(a^4)\,.$$

Thus, argue that the period of oscillations T is slowly changing with a.

(b) Use the Poincaré–Lindstedt method to show find

$$T(a) = 2\pi \left(1 + \frac{a^2}{16} + O(a^4)\right).$$

Note: This problem is degenerate, i.e., the perturbation does not select a specific periodic orbit. Yet, you can write out the periodicity conditions, expand in powers of a, and solve them.

2. (Verhulst, Exercise 11.4.) A satellite moves in the outer atmosphere of a spherically symmetric, homogeneous planet. The (nondimensionalized) equations of motion read

$$\ddot{\boldsymbol{r}} = -\frac{\boldsymbol{r}}{r^3} - \varepsilon\,\dot{\boldsymbol{r}}$$

where $\boldsymbol{r} = (x, y, z)$ and $r = \|\boldsymbol{r}\|$.

(a) Show that for given initial conditions, the motion takes place in a plane through the origin (in physical space, not phase space!). Thus, in the following, you can take $z = 0$ without loss of generality.

(b) Introduce polar coordinates $x = r\,\cos\theta$ and $y = r\,\sin\theta$ to find

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{1}{r^2} - \varepsilon\,\dot{r}\,,$$
$$2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = -\varepsilon\,r\,\dot{\theta}\,.$$

(c) Integrate the second equation to find

$$r^2\,\dot\theta = c\,\mathrm{e}^{-\varepsilon t}\,.$$

(The constant c is the initial angular momentum of the satellite.)

(d) Set $\rho = 1/r$ and use θ as a time-like variable. Show that this results in

$$\begin{aligned}\frac{\mathrm{d}^2\rho}{\mathrm{d}\theta^2} + \rho &= u\,,\\ \frac{\mathrm{d}u}{\mathrm{d}\theta} &= 2\,\varepsilon\,\frac{u^{3/2}}{\rho^2}\end{aligned}$$

with

$$u = \frac{1}{c^2}\,\mathrm{e}^{2\varepsilon t}\,.$$

(e) Obtain a Lagrange standard form by writing

$$\begin{aligned}\rho &= u + a(\theta)\,\cos\theta + b(\theta)\,\sin\theta\,,\\ \frac{\mathrm{d}\rho}{\mathrm{d}\theta} &= -a(\theta)\,\sin\theta + b(\theta)\,\cos\theta\,.\end{aligned}$$

(f) Apply averaging and give the approximations for $\rho(\theta)$ and $r(t)$ with an initially circular orbit where $\theta(0) = 0$, $r(0) = c^2$, and $\dot r(0) = 0$. Discuss the asymptotic validity of the approximation.