Differential Equations

Homework 5

Due in class Thursday, March 22, 2018

1. Consider the system

$$\dot{x} = 1 + y - x^2 - y^2 \,,$$
$$\dot{y} = 1 - x - x^2 - y^2 \,.$$

   (a) Determine the critical points and their character.

   (b) Show that the flow generated by this equation is symmetric with respect to the line $x + y = 0$.

   (c) Find an explicit periodic solution $\phi(t)$

   Hint: Write the equation in polar coordinates.

   (d) Derive the linearization of the equation about the periodic solution $\phi(t)$.

   (e) Determine both Floquet exponents of the linearized, time-periodic, system obtained in part (d).

   (f) Sketch the phase portrait of the system.

   Hint: Introduce new variables $u = x - y$ and $v = x + y$ that exploit the symmetry and find a first integral for the system in u and v coordinates (introduce $w = v^2$ and solve the equation for $\mathrm{d}w/\mathrm{d}w$ to obtain the first integral).

   (g) Show that $\phi(t)$ is stable, but not asymptotically stable.

   (h) Can you find a small perburbation to the system in the form

$$\dot{x} = 1 + y - x^2 - y^2 + \varepsilon\, g(x, y) \,,$$
$$\dot{y} = 1 - x - x^2 - y^2 + \varepsilon\, h(x, y) \,,$$

   which preserves the periodic orbit $\phi(t)$ and makes it asymptotically stable. Establish the asymptotic stability by computing the perturbed Floquet exponents.

2. (Verhulst, Exercise 8.4.) Consider the system

$$\dot{x} = 2\, y\, (z - 1) \,,$$
$$\dot{y} = -x\, (z - 1) \,,$$
$$\dot{z} = x\, y \,,$$

(a) Show that the trivial solution is stable.

(b) Is it asymptotically stable?

3. (Verhulst, Exercise 8.5.) Determine the stablity of the trivial solution of the system

$$\dot{x} = 2\,x\,y + x^3\,,$$
$$\dot{y} = x^2 - y^5\,.$$