Differential Equations

Homework 4

Due in class Tuesday, March 13, 2018

1. (From Verhulst, Exercise 4.8.) Consider the system

$$\dot{x} = \frac{\partial E}{\partial y} + \lambda\, E\, \frac{\partial E}{\partial x}\,,$$
$$\dot{y} = -\frac{\partial E}{\partial x} + \lambda\, E\, \frac{\partial E}{\partial y}$$

with $\lambda \in \mathbb{R}$ and

$$E(x,y) = y^2 - 2\,x^2 + x^4\,.$$

   (a) Put $\lambda = 0$. Determine the critical points and their character by linear analysis. What happens in the nonlinear system? Sketch the phase plane.

   (b) What happens to the critical points when $\lambda \neq 0$?

   (c) Choose $\lambda < 0$. Determine the ω-limit sets of the orbits starting at $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$, and $(1, 2)$.

2. (From Verhulst, Exercise 6.4.) For the equation

$$\dot{x} = Ax + B(t)x\,,$$

where A is a constant $n \times n$ matrix with all eigenvalues having strictly negative real part, and $B(t)$ is $n \times n$ matrix depending continuously on t. Show that there exists $\delta > 0$ such that if $\|B(t)\| \leq \delta$ for $t \geq 0$, then 0 is asymptotically stable.

3. Let $J = \lambda\, I + N$ be a Jordan block matrix of dimension k and eigenvalue λ. (Thus, N is the nilpotent matrix with 1 on the superdiagonal and 0 everywhere else.)

   Show that if f is an analytic function, $f(J)$ is well defined and

$$f(J) = \sum_{i=0}^{k} \frac{f^{(i)}(\lambda)}{i!}\, N^i\,.$$

4. Consider the differential equation

$$\dot{x} = A(t)x$$

with

$$A(t) = S(t)^{-1}BS(t)$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \qquad \text{and} \qquad S(t) = \begin{pmatrix} \cos(at) & -\sin(at) \\ \sin(at) & \cos(at) \end{pmatrix} .$$

(a) Show that, for any t, all eigenvalues of $A(t)$ have negative real part.

(b) Show, that for suitable choice of a, the differential equation has solutions with $\|x(t)\| \to \infty$ as $t \to \infty$.