# Differential Equations

Homework 2

Due in class Tuesday, February 20, 2018

1. (a) Prove that for an $n \times n$ matrix $A$,
$$\frac{d}{dt}\mathrm{e}^{At} = A\mathrm{e}^{At}$$
and conclude that the solution $x = x(t)$ to
$$\dot{x} = Ax\,, \qquad x(0) = x_0\,,$$
is given by
$$x(t) = \mathrm{e}^{At}x_0\,.$$
(b) Show that if two $n \times n$ matrices $A$ and $B$ commute, i.e., if $AB = BA$, then
$$B\mathrm{e}^{At} = \mathrm{e}^{At}B\,.$$
*Hint:* Verify that $B\mathrm{e}^{At}x_0$ and $\mathrm{e}^{At}Bx_0$ solve $\dot{x} = Ax$, then use uniqueness.
(c) Show that if $A$ and $B$ commute, then
$$\mathrm{e}^{(A+B)t} = \mathrm{e}^{At}\,\mathrm{e}^{Bt}\,.$$
*Hint:* Use the result from part (b) to show that both sides are one parameter groups of transformations that solve the same differential equation.
(d) Use (c) to solve the differential equation
$$\dot{x} = Mx \qquad \text{where } M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. Let $A \in \mathbb{R}^{n\times n}$ such that $\operatorname{Re}\lambda \leq \alpha$ for every eigenvalue $\lambda$ of $A$. Let $k$ be the dimension of the biggest Jordan block in the Jordan canonical form of $A$. Show that there exists a constant $C$ such that
$$\|\mathrm{e}^{At}\| \leq C\,\mathrm{e}^{\alpha t}\left(1 + t + \cdots + \frac{t^{k-1}}{(k-1)!}\right).$$
*Hint:* First prove the result for a single Jordan block, using the ideas from parts (c) and (d) of the previous question. Then argue for general $A$.

3. Consider the damped harmonic oscillator in the form

$$\dot{x} = Ax$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

(a) Show that $A$ satisfies the assumption of Problem 2 above.

(b) Show that $A$ is *not* strictly negative definite. (A matrix $A \in \mathbb{R}^{n\times n}$ is strictly negative definite if $x^T Ax < 0$ for all $x \in \mathbb{R}^n$.)

(c) Show that rate of change of the energy $E = x^T x$ is not strictly negative at all times.

(d) Give a physical interpretation of this fact. Moreover, argue that it is nonetheless consistent with the conclusion of Problem 2.

4. Find all equilibrium points, determine their stability properties, and try to sketch the phase portraits of the following differential equations:

(a) $\dot{x}_1 = 2x_1 - x_1^2 - x_1x_2$

$\dot{x}_2 = -x_2 + x_1x_2$

(b) $\ddot{y} + \dot{y} + y^3 = 0$