Differential Equations

Homework 1

Due in class Tuesday, February 13, 2018

1. Consider the differential equation

$$\dot{y} = \sqrt{y}\,,$$
$$y(0) = 0\,.$$

   (a) Show that the solution is not unique.

   (b) Show that the vector field does not satisfy the assumptions of the Picard-Lindelöf theorem.

2. Suppose that $f \in C(\mathbb{R}^n \times \mathbb{R}_+)$ satisfies a *quasi-Lipshitz condition* in the sense that there exists $L > 0$ such that

$$\|f(x,t) - f(y,t)\| \leq L\,\|x-y\|\,\bigl|\ln\|x-y\|\bigr|$$

   for all $t \geq 0$ and $x, y \in \mathbb{R}^n$ with $\|x - y\| \leq \frac{1}{2}$. Write $x(t)$ and $y(t)$ to denote two solutions of the differential equation

$$\dot{x} = f(x,t)\,.$$

   Show that

$$\|x(t) - y(t)\|^2 \leq \|x(0) - y(0)\|^2 \, \exp(\exp(ct))\,.$$

   (What is c?)

   Remark: Local existence of solutions under the quasi-Lipshitz condition can be shown by a modification of the Picard–Lindelöf argument which is slightly more involved than the argument required here. I pose this as a challenge problem.

3. Consider the Volterra–Lotka system, here with all coefficients set to one,

$$\dot{x} = x - x\,y\,,$$
$$\dot{y} = x\,y - y\,.$$

   (a) Show that when $x > 0$ and $y > 0$ at time $t = 0$, the solutions remain strictly positive for as long as they exist.

(b) Show that positive solutions exist for all times.

(c) What can you say when you drop the condition of positivity of the initial values?

4. (From Verhulst, p. 23.) Consider the equation

$$\ddot{x} - \lambda\,\dot{x} - (\lambda - 1)\,(\lambda - 2)\,x = 0$$

with λ a real parameter. Find the critical points and characterize these points. Sketch the flow in the phase plane and indicate the direction of the flow.