1. Consider the system

$$\dot{x} = -x + y^2\,,$$
$$\dot{y} = -y^3 + x^2\,.$$

(a) Find all critical points and determine their linear stability.

(b) Clearly, the origin is a critical point where linear stability analysis is inconclusive. Show that the origin is an asymptotically stable equilibrium point.
Hint: Show that $V(x,y) = x^2\,(1+y) + y^2$ is a Lyapunov function in some neighborhood of the origin.
You may find the following elementary inequality useful:

$$a\,b \le \frac{\delta}{2}\,a^2 + \frac{1}{2\delta}\,b^2$$

for all $a, b \in \mathbb{R}$ and $\delta > 0$.

(c) Sketch the phase portrait.

(10+10+5)

(a) $-x + y^2 = 0$
$-y^3 + x^2 = 0$

$\Rightarrow x = y = 0$ or $x = y^2$ so that $y^3 = y^4 \Rightarrow y = 1,\ x = 1$

$$DF = \begin{pmatrix} -1 & 2y \\ 2x & -3y \end{pmatrix}$$

@(0,0): $DF = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ so that $\lambda_1 = -1$, $\lambda_2 = 0$

$\Rightarrow$ (0,0) is linearly neutrally stable

@(1,1): $DF = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

$\det(DF - \lambda I) = (1+\lambda)(3+\lambda) - 4 = \lambda^2 + 4\lambda - 1$

$\Rightarrow \lambda_\pm = -2 \pm \sqrt{4+1} = -2 \pm \sqrt{5}$

$\Rightarrow$ (1,1) is a saddle.

(b) $$V = x^2 + y^2 + x^2 y$$

$$\geq x^2 + y^2 - \tfrac{1}{2}x^4 - \tfrac{1}{2}y^2$$

$$\geq \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 \qquad \text{if } |x| \leq 1$$

$\Rightarrow$ V is pos. def. for $|x| \leq 1$.

$$\dot V = 2x\dot x\,(1+y) + x^2\dot y + 2y\dot y$$

$$= 2x(-x+y^2)(1+y) + (x^2+2y)(-y^3+x^2)$$

$$= -2x^2 + \underbrace{2xy^2}_{\leq x^2+y^4} - 2x^2y + 2xy^3 - y^3x^2 + x^4 - 2y^4 + 2yx^2$$

$$\leq -x^2 - y^4 + x^4 + \underbrace{2xy^3}_{\leq \frac{1}{2}x^2 + 2y^6} - \underbrace{y^3x^2}_{\leq \frac{1}{2}x^4 + \frac{1}{2}y^6}$$

$$\leq -\tfrac{1}{2}x^2 - y^4 + \tfrac{3}{2}x^4 + \tfrac{5}{2}y^6$$

So for x, y sufficiently small, $\dot V$ is neg. def.

$\Rightarrow$ $(0,0)$ is asymptotically stable

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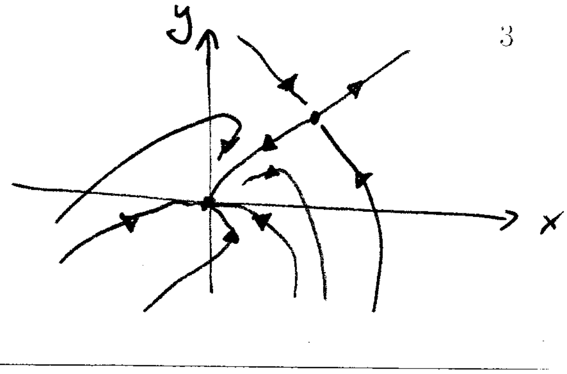
(c)



(Cf. Verhulst, p. 35, Example 33)

2. Consider the n-dimensional linear system

$$\dot{x} = A(t)x$$

where $A(t)$ is T-periodic. Recall that the Floquet theorem states that a fundamental matrix solution to this equation can be written as

$$\Phi(t) = P(t)\,e^{Bt}$$

where $P(t)$ is a T-periodic and B a constant $n \times n$ matrix.

(a) Consider the case $n = 1$. Determine B and give necessary and sufficient conditions so that every solution $x(t)$ remains bounded as $t \to \infty$.
Hint: Introduce the average

$$\overline{A} = \frac{1}{T}\int_0^T A(s)\,ds\,.$$

(b) Consider the case $n = 2$ with

$$A(t) = g(t)\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $g(t)$ is a T-periodic real-valued function. Determine B and give necessary and sufficient conditions so that every solution $x(t)$ remains bounded as $t \to \infty$.

(10+10)

(a) $\frac{d}{dt}\left(e^{-\int_0^t A(\tau)\,d\tau}\, x(t)\right) = 0 \quad \Rightarrow \quad x(t) = e^{\int_0^t A(\tau)\,d\tau}\, x(0)$

$$\Rightarrow x(t) = \underbrace{e^{\int_0^t (A(\tau)-\overline{A})\,d\tau}}_{= P(t)}\ \underbrace{e^{\overline{A}t}}_{= e^{Bt}}\ x(0)$$

(Clearly, $P(t)$ is T-periodic!)

Thus, solutions remain bounded as $t \to \infty$ iff $B = \overline{A} \le 0$.

(b) Due to the special structure of A, solution by integrating factor as in (a) still works, so

$$x(t) = e^{\int_0^t (g(\tau) - \bar{g})\, d\tau\, J} \; e^{\bar{g} J t} \; x(0) .$$

Thus, when $\mathrm{Re}[\bar{g}\lambda] < 0$, $\begin{pmatrix}0\\0\end{pmatrix}$ is asymptotically stable, i.e., solutions remain bounded.

When $\mathrm{Re}[\bar{g}\lambda] > 0$, all solutions grow exponentially.

When $\mathrm{Re}[\bar{g}\lambda] = 0$, $\bar{g} \neq 0$, $\bar{g}J$ has a 2-dimensional Jordan block, hence a 1-dimensional subspace of secular growth.

$\Rightarrow$ $\mathrm{Re}[\bar{g}\lambda] < 0$ or $\bar{g} = 0$ is necessary and sufficient for all solutions to remain bounded.

3. Consider the system

$$\dot{x} = y\,,$$
$$\dot{y} = -x + y\,(1 - x^2 - 2\,y^2)\,.$$

Show that the system has at least one periodic orbit. (10)

Set $R = \frac{1}{2}(x^2 + y^2)$

$\Rightarrow \quad \dot{R} = x\dot{x} + y\dot{y} = xy - yx + y^2(1 - x^2 - 2y^2)$

$\leq y^2(1-2R)$

Moreover, the only critical point is $(0,0)$, where

$DF = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\Rightarrow$ $(0,0)$ is a repelling focus.

Thus, the annulus $\varepsilon \leq R \leq \frac{1}{2}$ for $\varepsilon > 0$ sufficiently small is positively invariant and does not contain critical points.

Thus, by Poincaré-Bendixon, it must contain at least one periodic orbit.

4. Consider an autonomous planar differential equation.

(a) Suppose a point x is not on a periodic orbit. Show that a period orbit cannot be at the same time the ω-limit set and the α-limit set of the orbit through x.

(b) Show that the region bounded by a periodic orbit must contain an equilibrium point.

Hint: Argue by contradiction and use part (a).

(5+5)

(a) Take a point p on the periodic orbit and Σ a transversal through p. Then, by a lemma from class, the points of intersection of $\phi_t x$ with Σ are monotonically ordered. Thus, the periodic orbit cannot be α-limit and ω-limit set at the same time.

(b) Let P denote the set of all periodic orbits contained within and including the given periodic orbit p_0. Let $S(p)$ denote the interior of the set bounded by periodic orbit p.

Let

$$A = \bigcap_{p \in P} S(p).$$

If $A = \{\}$, then $\exists \{p_i\} \subset P$ nested s.t. $\bigcap_{i=1}^{\infty} \overline{S(p_i)}$ is a single point, which must be an equilibrium.

If $A \neq \{\}$, take $x \in A$. 7 Since there is no periodic orbit in A, by (a), either the α-limit or the ω-limit set of x must be an equilibrium point.

5. Suppose $x(t)$ is a solution to the n-dimensional differential equation

$$\dot{x} = f(x) + \varepsilon\, g(x)$$

where $f, g \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. Further, let $y(t)$ solve the equation

$$\dot{y} = f(y)$$

with $y(0) = x(0)$.

(a) Show that $x(t) - y(t) = O(\varepsilon)$ on the time scale 1, assuming that both solutions exist on this time scale.

(b) Suppose you know that the solution $y(t)$ is bounded (therefore exists) on some time interval $[0, T]$. What can you say about the time interval of existence for $x(t)$ as $\varepsilon \to 0$?

(10+5)

(a) Set $w = x - y$. Then

$$\dot{w} = f(x) - f(y) + \varepsilon g(x)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|w\|^2 = w \cdot \big(f(x) - f(y)\big) + \varepsilon\, w \cdot g(x)$$

$$\leq \|w\| \cdot \Big(\underbrace{\sup_{\xi \in B} \|Df(\xi)\|}_{\leq C} \|w\| + \varepsilon \underbrace{\sup_{\xi \in B} \|g(\xi)\|}_{\leq M} \Big)$$

where B is some ball containing both trajectories. It exists, at least for some interval of time $[0,T]$, by the Picard-Lindelöf Theorem.

$$\Rightarrow \frac{d}{dt}\|w\| \leq c\,\|w\| + \varepsilon M$$

This differential inequality is easily integrated using an integrating factor (or directly via Gronwall's lemma):

$$\|w(t)\| \leq e^{ct} \underbrace{\|w(0)\|}_{=0} + \frac{\varepsilon M}{c}\left(e^{ct}-1\right)$$

$$\leq \varepsilon \frac{M}{c} e^{cT}$$

which is the statement claimed.

(b) Let $B = B_{2R}$ with $R = \max_{t\in[0,T]} \|y(t)\|$, and run the argument above with this definition of the ball. This can be done up to some time $T^* \leq T$ when $x(t)$ leaves B. However, this can only happen if $\|w\| > R$, i.e.

$$R < \varepsilon \frac{M}{c} e^{cT^*} \leq \varepsilon \frac{M}{c} e^{cT}$$

Thus, for $\varepsilon < \varepsilon^* = \frac{Rc}{M} e^{-cT}$, $x(t)$ remains in B, so, in particular, it exist on the entire interval $[0,T]$.