Applied Differential Equations and Modeling

Homework 7

Due in class Thursday, May 2

1. Find the Laplace transform of the given function.

   (a) $f(t) = t^{10}$

   (b) $f(t) = \mathrm{e}^{2t} \cos 3t$

   (c) $f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t > 1 \end{cases}$

   (d) $f(t) = t^n \, \mathrm{e}^{at}$

2. Show that the Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L} \int_0^t f(\tau) \, \mathrm{d}\tau = \frac{1}{s} \, \mathcal{L}(f)$$

   for all functions f such that the transforms on the left and on the right hand sides are well defined.

3. Compute the inverse Laplace transform of the given function.

   (a) $F(s) = \dfrac{2}{s^2 + 3s - 4}$

   (b) $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$

   (c) $F(s) = \dfrac{s^3 - 2s^2 - 6s - 6}{(s^2 + 2s + 2) \, s^2}$

4. Use the Laplace transform to solve the given initial value problem.

   (a) $y'' - y' - 6y = 0$
   with $y(0) = 1$, $y'(0) = -1$

   (b) $y'' + \omega^2 \, y = \cos 2t$
   for $\omega^2 \neq 4$ with $y(0) = 1$, $y'(0) = 0$

(c) $y'''' - 4y = 0$
with $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 0$

5. Find the Laplace transform of the given periodic function.

   (a) $f(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ -1 & \text{for } 1 \le t < 2 \end{cases}$
   and f has period 2.

   (b) $f(t) = t$ for $0 \le t < 1$ and f has period 1.

6. For a function $f(t)$, write $F(s)$ to denote the Laplace transform. Prove the following.

   (a) $\mathcal{L}(e^{ct} f(t)) = F(s - c)$

   (b) $\mathcal{L}(f(ct)) = \dfrac{1}{c} F\left(\dfrac{s}{c}\right)$