Applied Differential Equations and Modeling

Homework 2

Due in class Tuesday, February 20, 2018

1. Consider the differential equation

$$y' = (1 - 2x)/y\,, \quad y(1) = -2\,.$$

   Solve this initial value problem, plot the graph of the solution, and state on which interval the solution is defined.

2. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - 4x}{x - y}\,.$$

   (a) Show that, setting $v = x/y$, that this differential equation can be written in the form

$$v + x\,\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{1 - v}\,.$$

   (b) Show that this new form of the equation is separable, and solve the equation. Write, as your final answer, y as a function of x.

   (c) Draw the direction field in the x-y-plane, and a few integral curves (i.e., solution curves to the differential equation which you found in part b).

   Remark: Note that the computation above showed that the direction field depends only on y/x; this should be clearly seen in your answer. Such equations are sometimes called *homogeneous*.

3. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from height h.

   (a) Show that $v = \sqrt{2gh}$, where g is the constant of gravitational acceleration.

   (b) By equating the rate of outflow to the rate of change of volume in the tank, show that $h(t)$ satisfies the differential equation

$$A(h)\,\frac{\mathrm{d}h}{\mathrm{d}t} = -\alpha\, a\,\sqrt{2gh}\,,$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller that a. The value of α for water is about 0.6.

(c) Consider a water tank in the form of right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m and the radius of the circular outlet is $\frac{1}{10}$ m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

4. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from a position 30 m above ground. The ball is subject to gravitational force and a force due to to air resistance of magnitude $k\,|v|$ with $k = 1/30\,\text{kg/s}$.

   (a) Find the maximum height above the ground that the ball reaches.

   (b) Find the time that the ball hits the ground.

   (c) Plot the graphs of velocity and position versus time.

5. Solve the initial value problem

$$y' = 2ty^2\,, \quad y(0) = y_0$$

and determine how the interval on which the solution exists depends on y_0.