1. Solve the differential equation

$$t\,y' + 2\,y = \sin t\,, \qquad y(\pi/2) = 1\,.$$

(10)

Standard form:

$$y' + \frac{2}{t}\,y = \frac{1}{t}\sin t$$

Integrating factor: $M = e^{\int_1^t \frac{2}{\tau}d\tau} = e^{2\ln\tau\big|_{\tau=1}^{\tau=t}} = e^{2(\ln t - \ln 1)}$

$$= t^2$$

$$\Rightarrow \frac{d}{dt}\left(t^2 y\right) = t\sin t$$

$$\Rightarrow t^2 y\Big|_{\frac{\pi}{2}}^{t} = \int_{\frac{\pi}{2}}^{t} t\sin t\,dt$$

$$= \underbrace{-t\cos t\Big|_{\frac{\pi}{2}}^{t}}_{=-t\cos t} + \underbrace{\int_{\frac{\pi}{2}}^{t}\cos t\,dt}_{=\sin t\big|_{\frac{\pi}{2}}^{t} = \sin t - 1}$$

$$= -t\cos t + \sin t - 1$$

$$\Rightarrow t^2 y(t) = \left(\frac{\pi}{2}\right)^2 - t\cos t + \sin t - 1$$

$$\Rightarrow y(t) = \frac{1}{t^2}\left(\left(\frac{\pi}{2}\right)^2 - t\cos t + \sin t - 1\right)$$

2. Consider the differential equation

$$y' = 2t\,y^2 .$$

(a) Solve the equation with initial condition $y(0) = a$. (Note: the case $a = 0$ is special and requires separate discussion.)

(b) For which values of a does the solution exist for all $t \geq 0$? If the solution fails to exist, state the interval of existence.

(5+5)

(a) $$\frac{dy}{y^2} = 2t\,dt \quad \Rightarrow \quad \int_a^{y(t)} \frac{dy}{y^2} = \int_0^t 2t\,dt = t^2$$

$$\Rightarrow \; -y^{-1} \Big|_a^{y(t)} = t^2 \quad \Rightarrow \quad \frac{1}{a} - \frac{1}{y(t)} = t^2$$

$$\Rightarrow \; y(t) = \frac{1}{\frac{1}{a} - t^2} \qquad \text{if } a \neq 0 \; !$$

If $a=0$, $y(t)=0$ is obviously a solution.

(b) The solution ceases to exist when $\frac{1}{a} = t^2 \Rightarrow t = \sqrt{\frac{1}{a}}$

Thus, when $a \leq 0$, the solution exists for all $t \geq 0$,

when $a > 0$, the solution exists for $t \in [0, \sqrt{\frac{1}{a}})$.

3. Consider the differential equation

$$y' = y(1 - y^2).$$

(a) Find all equilibrium points of the equation.

(b) Classify each equilibrium point as stable or unstable.

(c) Indicate the equilibrium points in a t-y graph and sketch several other solutions *without* solving the equation.

(5+5+5)

(a) $y(1-y^2) = 0$ if $y = 0$ or $y = \pm 1$.

(b) at $y = -1$, $f(y) = y(1-y^2)$ changes from + to −

$\Rightarrow$ stable equilibrium point

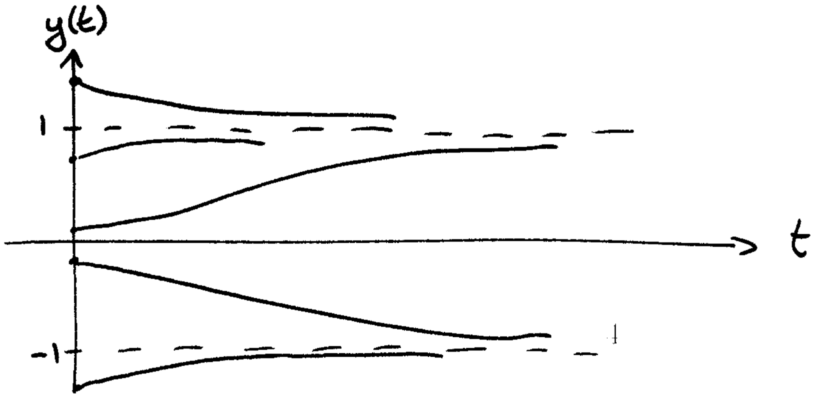
at $y = 0$, $f(y)$ changes from − to +

$\Rightarrow$ unstable equilibrium point

at $y = 1$, $f(y)$ changes from + to −

$\Rightarrow$ stable equilibrium point

(c)



4. Let $P(t)$ denote the number of fish in a lake at time t, and let C denote the "carrying capacity" of the lake. Suppose further that fishermen catch fish at a constant "harvest rate" h. (Thus, different from the homework problem on the same topic, the fishermen increase their effort if the number of fish in the lake gets small!) Then the population of fish satisfies the equation

$$\frac{dP}{dt} = \left(1 - \frac{P}{C}\right) P - h \qquad \text{with} \qquad P(0) = P_0 .$$

In the following, we assume that $h < C/4$.

(a) Show that the equation has two positive equilibrium points, the larger one stable, the smaller one unstable.

(b) Indicate the equilibrium points in a t-P graph and sketch several other solutions *without* solving the equation.

(c) How does the population of fish behave long-time? Discuss, in particular, the possibility of extinction. Write a recommendation to the fishermen.

(d) *Extra credit:* Show that, if $h > C/4$, the population will become extinct in finite time no matter how large the initial population of fish.

(5+5+5+5)

(a) $\left(1-\frac{P}{C}\right)P - h = 0 \quad \Rightarrow \quad P^2 - CP + Ch = 0$

quadratic formula: $P_{1,2} = \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - Ch}$.

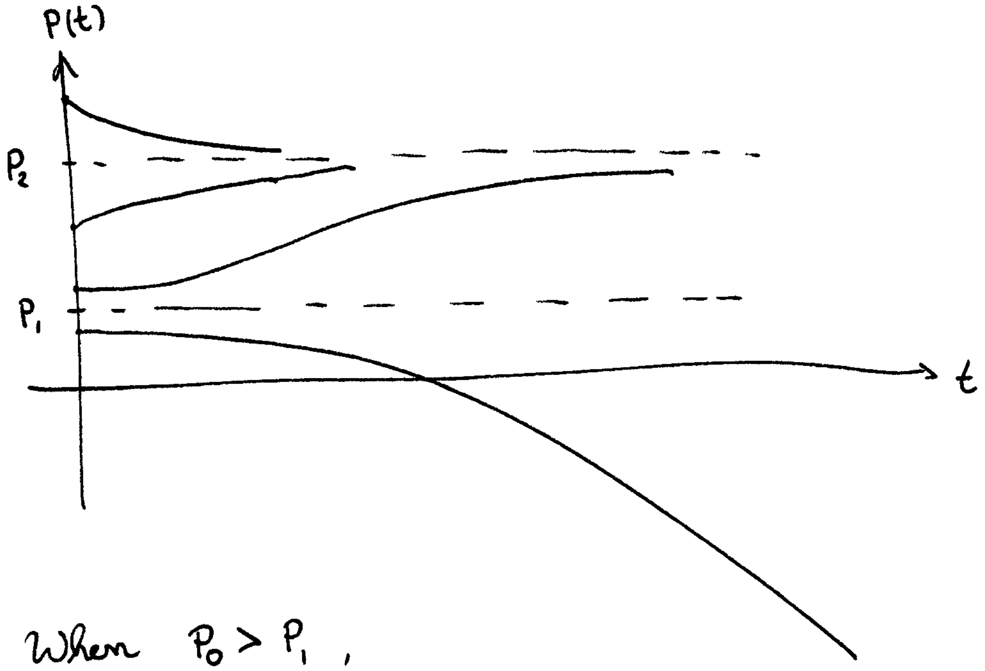
So if $\frac{C^2}{4} > Ch \Rightarrow h < \frac{C}{4}$, there are two equ. points.

Clearly, $\left(1-\frac{P}{C}\right)P - h$ changes sign from $-$ to $+$ to $-$, so the smaller eq. point is unstable, the larger is stable.

Since $\sqrt{\frac{C^2}{4} - Ch} < \frac{C}{2}$, both equilibrium points

are positive.

(b)



(c) When $P_0 > P_1$,

$$\lim_{t\to\infty} P(t) = P_2$$

When $P_0 < P_1$, $P(t)$ becomes zero in a finite time, then the population becomes extinct.

Recommendation (Version A):

Make sure that $P_0 > P_1 \Rightarrow P_0 > \frac{C}{2} - \sqrt{\left(\frac{C}{2}\right)^2 - Ch}$

$$\Rightarrow \left(\frac{C}{2}\right)^2 - Ch > \left(\frac{C}{2} - P_0\right)^2 = \left(\frac{C}{2}\right)^2 - CP_0 + P_0^2$$

$$\Rightarrow h < P_0 - \frac{P_0^2}{C}$$

The rate of fishing must not exceed $P_0\left(1 - \frac{P_0}{C}\right)$.

Recommendation (Version B):

(The fishermen might not know the value of P_0 and C...)
See if the number of fish is decreasing. If it is, and the rate of decrease is getting bigger, you are fishing too much.

(d) Write $h = \frac{C}{4} + z$ with $z > 0$.

Then
$$\frac{dP}{dt} = \left(1 - \frac{P}{C}\right)P - \frac{C}{4} - z$$

$$= -\frac{1}{C}\underbrace{\left(P - \frac{C}{2}\right)^2}_{\geq 0} - z$$

$$\leq -z$$

$$\Rightarrow \quad P(t) \leq P_0 - zt$$

So the population becomes extinct no later than

$$t = \frac{P_0}{z}.$$

5. Consider the second order differential equation

$$y'' + y' - 2y = 0.$$

(a) Write this equation as a system of two first-order equations in matrix form with matrix A.

(b) Compute the eigenvalues of A.

(c) Compute the eigenvectors of A.

(d) Write out the general solution $y(t)$ for the second order equation.

(e) Write out the solution with initial condition $y(0) = 0$ and $y'(0) = 1$.

(f) Sketch the qualitative behavior of the equation in the y-y' phase plane.

(5+5+5+5+5+5)

(a) $\left.\begin{array}{l} x_1' = x_2 \\ x_2' = -x_2 + 2x_1 \end{array}\right\} \quad x' = Ax \quad$ with $\quad A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

(b) $-\lambda(-1-\lambda) - 2\cdot 1 = 0 \quad \Rightarrow \quad \lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda_1 = -2 \;, \; \lambda_2 = 1$

(c) For λ_1: $\quad A - \lambda_1 I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} v_1 = 0 \quad$ is solved by $\quad v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For λ_2: $\quad A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} v_2 = 0 \quad$ is solved by $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-2t} + c_2 e^{t}$

(e) $y'(t) = -2c_1 e^{-2t} + c_2 e^{t}$

So the initial conditions imply:

$$\left.\begin{aligned} C_1 + C_2 &= 0 \\ -2C_1 + C_2 &= 1 \end{aligned}\right\} \Rightarrow \quad 3C_1 = -1 \quad \Rightarrow \quad C_1 = -\frac{1}{3}$$

$$\Rightarrow \quad C_2 = \frac{1}{3}$$

$$\Rightarrow \quad y(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

(d)