Applied Differential Equations and Modeling

Final Exam

May 20, 2018

1. Solve the differential equation
$$2\,y' + t\,y = 2\,, \qquad y(0) = 1\,.$$
(10)

2. (a) Solve the differential equation
$$y' = (1-2t)\,y^2\,, \qquad y(0) = -1\,.$$
   (b) For which interval of time does the solution exist?

(5+5)

3. Consider the differential equation
$$y' = y^2 - y\,.$$
   (a) Find all equilibrium points of the equation.
   (b) Classify each equilibrium point as stable or unstable.
   (c) Indicate the equilibrium points in a t-y graph and sketch several other solutions *without* solving the equation.
   (d) For which values of $y(0)$ does the solution exist for all positive times? (Argue, if possible, without solving the equation.)

(5+5+5+5)

4. (a) Compute, without using the table of Laplace transforms, the Laplace transform of $f(t) = u(t-1)$, where u is the unit step function.
   (b) Find the inverse Laplace transform of
$$F(s) = \frac{s}{(s-1)^2+1}\,.$$
   You may use the table of Laplace transforms.

(5+5)

5. Verify the following property of the Laplace transform:

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)\,.$$

(10)

6. (a) Use the Laplace transform to solve the equation

$$y'' + y = \delta(t)\,, \qquad y(0) = y'(0) = 0\,.$$

(b) Use the Laplace transform to solve the equation

$$y'' + y = \delta(t - 2\pi)\,, \qquad y(0) = y'(0) = 0\,.$$

(c) What happens for

$$y'' + y = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \delta(t - 6\pi) + \dots\,,$$

again with $y(0) = y'(0) = 0$? Describe the features of the solution *in words*, using technical terms when applicable. (No formula required, but permitted.)

(10+5+5)

7. Consider the second order differential equation

$$y'' + 2\,y' + y = g(t)\,.$$

(a) Write this equation as a system of two first-order equations in matrix form with matrix A.

(b) Compute the eigenvalues of A.

(c) Compute the eigenvector(s) and, if applicable, generalized eigenvector of A.

(d) Write out the general solution $\boldsymbol{x}(t)$ for the *homogeneous* (the case when $g(t) = 0$) first order system from part (a).

(e) Write out the general solution $y(t)$ for the given *homogeneous* second order equation.

(f) Sketch the qualitative behavior of the homogeneous equation in the y-y' phase plane.

(g) Use the method of undetermined coefficients to find a particular solution when $g(t) = \cos t$.

(h) Continuing the problem from (g), write out the solution with initial condition $y(0) = 0$ and $y'(0) = 1$.

(i) Re-derive your answer to part (h) using the Laplace transform.

(j) What is the impulse response function of this system?

(k) Is the system BIBO-stable? Show your computation.

(l) What is the equation a model of? Describe in words.

(5 points each)