1. Solve the differential equation

$$2y' + ty = 2, \qquad y(0) = 1.$$

(10)

The integrating factor is $M = e^{\frac{1}{2}\int_0^t t\,dt} = e^{t^2/4}$.

$$\Rightarrow \quad \frac{d}{dt}\left(e^{t^2/4} y\right) = e^{t^2/4}$$

$$\Rightarrow \quad e^{t^2/4} y(t) \Big|_0^t = \int_0^t e^{\tau^2/4}\, d\tau$$

$$\Rightarrow \quad y(t) = e^{-t^2/4} y(0) + e^{-t^2/4} \int_0^t e^{\tau^2/4}\, d\tau$$

$$= e^{-t^2/4} + e^{-t^2/4} \int_0^t e^{\tau^2/4}\, d\tau$$

The integral cannot be evaluated in closed form, so this is the final answer.

2. (a) Solve the differential equation

$$y' = (1-2t)\,y^2\,, \qquad y(0) = -1\,.$$

(b) For which interval of time does the solution exist?

(5+5)

(a) $\dfrac{dy}{y^2} = (1-2t)\,dt$

$$\Rightarrow \int_{-1}^{y(t)} \frac{dy}{y^2} = \int_0^t (1-2t)\,dt$$

$$\Rightarrow -\frac{1}{y}\Bigg|_{-1}^{y(t)} = t - t^2$$

$$\Rightarrow -\frac{1}{y(t)} - 1 = t - t^2$$

$$\Rightarrow \frac{1}{y(t)} = -1 - t + t^2$$

$$\Rightarrow y(t) = \frac{1}{t^2 - t - 1}$$

(b) The solution has a vertical asymptote when

$$t = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \pm \frac{\sqrt{5}}{2},$$

so the solution of the initial value problem exists for $-1 \leq t < \frac{1}{2} + \frac{\sqrt{5}}{2}$. (Or, if allowing backward evolution, on the interval $\left(\frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$.)

3. Consider the differential equation

$$y' = y^2 - y\,.$$

(a) Find all equilibrium points of the equation.

(b) Classify each equilibrium point as stable or unstable.

(c) Indicate the equilibrium points in a t-y graph and sketch several other solutions *without* solving the equation.

(d) For which values of $y(0)$ does the solution exist for all positive times? (Argue, if possible, without solving the equation.)

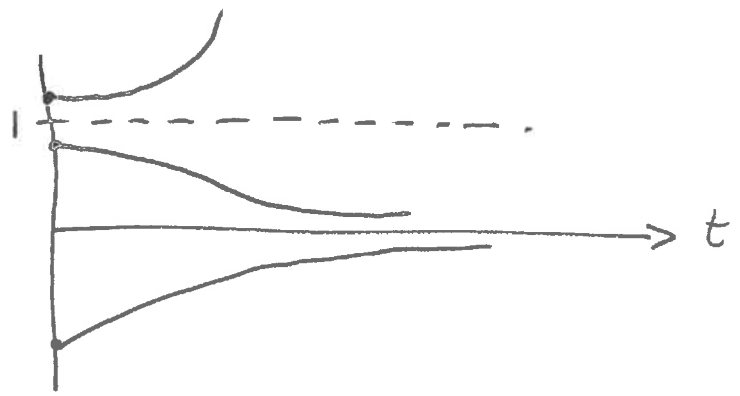
(5+5+5+5)

(a) $y^2 - y = 0 \Rightarrow y = 0$ or $y = 1$

(b) At $y=0$, RHS changes from + to − $\Rightarrow$ it is stable

$y=1$, RHS " " − to + $\Rightarrow$ it is unstable

(c)



(d) We know that the equation $y' = y^2$ has blow-up in finite time (example from class). Since, for y large enough, the term y^2 dominates (e.g. if $y \geq 2$ then $y^2 - y \geq \frac{1}{2} y^2$ so that $y' \geq \frac{1}{2} y^2$ which also blows up), the solution exists for all $t \geq 0$ if and only if $y(0) \leq 1$, i.e., is in or at the boundary of the stable region.

4. (a) Compute, without using the table of Laplace transforms, the Laplace transform of $f(t) = u(t-1)$, where u is the unit step function.

(b) Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s-1)^2 + 1}.$$

You may use the table of Laplace transforms.

(5+5)

(a) $F(S) = \int_0^\infty e^{-st} \, u(t-1) \, dt$

$= \int_1^\infty e^{-st} \, dt = \frac{1}{-S} e^{-st} \Big|_1^\infty = \frac{e^{-S}}{S}$

(b) $F(S) = \frac{S-1}{(S-1)^2+1} + \frac{1}{(S-1)^2+1}$

$\downarrow (23) \qquad \downarrow (22)$

$\Rightarrow f(S) = e^{t} \cos t + e^{t} \sin t$

5. Verify the following property of the Laplace transform:

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)\,.$$

(10)

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st}\, f'(t)\, dt$$

$$\overset{\text{I.b.p.}}{=} e^{-st}\, f(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st}\, f(t)\, dt$$

$$\overset{*}{=} -f(0) + s\int_0^\infty e^{-st}\, f(t)\, dt$$

$$= s\,\mathcal{L}[f(t)] - f(0)$$

* Under suitable decay assumptions on f.

6. (a) Use the Laplace transform to solve the equation

$$y'' + y = \delta(t), \qquad y(0) = y'(0) = 0.$$

(b) Use the Laplace transform to solve the equation

$$y'' + y = \delta(t - 2\pi), \qquad y(0) = y'(0) = 0.$$

(c) What happens for

$$y'' + y = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \delta(t - 6\pi) + \dots,$$

again with $y(0) = y'(0) = 0$? Describe the features of the solution *in words*, using technical terms when applicable. (No formula required, but permitted.)

(10+5+5)

(a) $s^2 Y(s) + Y(s) = 1$

$\Rightarrow Y(s) = \dfrac{1}{s^2+1}$

$\Rightarrow y(t) = \sin t$ by formula (13) for $t \geq 0$

(b) $s^2 Y(s) + Y(s) = e^{-2\pi s}$

$\Rightarrow Y(s) = e^{-2\pi s} \dfrac{1}{s^2+1}$

$\Rightarrow y(t) = u(t-2\pi) \sin(t-2\pi) = u(t-2\pi) \sin t$

(c) Following the same pattern, and using the superposition principle,

$y(t) = \sin(t) \left(u(t) + u(t-2\pi) + u(t-4\pi) + \dots \right)$

i.e., it's a sin-function whose amplitude is an infinite staircase. This is an example of resonance.

7. Consider the second order differential equation

$$y'' + 2\,y' + y = g(t)\,.$$

(a) Write this equation as a system of two first-order equations in matrix form with matrix A.

(b) Compute the eigenvalues of A.

(c) Compute the eigenvector(s) and, if applicable, generalized eigenvector of A.

(d) Write out the general solution $\boldsymbol{x}(t)$ for the *homogeneous* (the case when $g(t) = 0$) first order system from part (a).

(e) Write out the general solution $y(t)$ for the given *homogeneous* second order equation.

(f) Sketch the qualitative behavior of the homogeneous equation in the y-y' phase plane.

(g) Use the method of undetermined coefficients to find a particular solution when $g(t) = \cos t$.

(h) Continuing the problem from (g), write out the solution with initial condition $y(0) = 0$ and $y'(0) = 1$.

(i) Re-derive your answer to part (h) using the Laplace transform.

(j) What is the impulse response function of this system?

(k) Is the system BIBO-stable? Show your computation.

(l) What is the equation a model of? Describe in words.

(5 points each)

Answer:

(a) $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f}$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\boldsymbol{f} = \begin{pmatrix} 0 \\ g \end{pmatrix}$.

(b) The characteristic equation $\det(A - \lambda I) = 0$ reads $\lambda\,(2 + \lambda) + 1 = 0$, which has a double root $\lambda = -1$.

(c) $(A - \lambda I)\boldsymbol{v} = 0$ is solved by $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since there is no second linearly independent eigenvector, there must be a generalized eigenvector which solves $(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$. The choice $\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a possible generalized eigenvector.

(d) $\boldsymbol{x}(t) = c_1\,\boldsymbol{v}\,\mathrm{e}^{-t} + c_2\,(\boldsymbol{v}\,t + \boldsymbol{w})\,\mathrm{e}^{-t}$

(e) $y_h(t) = c_1\,\mathrm{e}^{-t} + c_2\,t\,\mathrm{e}^{-t}$

(f) See separate sheet.

(g) Let's make the ansatz $y(t) = A \sin t + B \cos t$. Then $y'(t) = A \cos t - B \sin t$ and $y'' = -A \sin t - B \cos t$. This leads to the system of linear equations $-A - 2B + A = 0$ and $-B + 2A + B = 1$, so that $B = 0$ and $A = \frac{1}{2}$. So the particular solution is $y_p(t) = \frac{1}{2} \sin t$.

(h) The solution is of the form $y(t) = y_h(t) + y_(t)$. From (g), we have $y_p(0) = 0$ and $y_p'(0) = \frac{1}{2}$. So we need $y_h(0) = 0$ and $y_h'(0) = \frac{1}{2}$ so that y satisfies the given initial conditions. From (c) or (d): $y_h'(t) = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t \mathrm{e}^{-t}$. Thus, $y_h(0) = c_1$ and $y_h'(0) = -c_1 + c_2$. We conclude that $c_1 = 0$ and $c_2 = \frac{1}{2}$. Altogether,

$$y(t) = \tfrac{1}{2} t \mathrm{e}^{-t} + \tfrac{1}{2} \sin t .$$

(i) Taking the Laplace transform of the equation, we find that

$$s^2 Y(s) - 1 + 2 s Y(s) + Y(s) = \frac{s}{s^2 + 1},$$

so that

$$Y(s) = \frac{1}{(s+1)^2} + \frac{s}{(s+1)^2(s^2+1)} .$$

Noting the partial fraction decomposition

$$\frac{s}{(s+1)^2(s^2+1)} = -\frac{1}{2} \frac{1}{(1+s)^2} + \frac{1}{2} \frac{1}{1+s^2},$$

we find that

$$Y(s) = \tfrac{1}{2} \frac{1}{(s+1)^2} + \frac{1}{2} \frac{1}{1+s^2} .$$

Using formulas (20) and (13) from the table of Laplace transforms, we obtain, again,

$$y(t) = \tfrac{1}{2} t \mathrm{e}^{-t} + \tfrac{1}{2} \sin t .$$

(j) The impulse response is the solution to the equation for $g(t) = \delta(t)$ so that $G(s) = 1$, and all initial values are zero. Thus, it has Laplace transform

$$H(s) = \frac{1}{(s+1)^2}$$

so that

$$h(t) = t \mathrm{e}^{-t} .$$

(k) Since the Laplace transform of the impulse response, the transfer function, is proper, the system is BIBO-stable if and only if the roots of the denominator polynomial have negative real part. Clearly, $H(s)$ has a double root at $s = -1$, so it is BIBO-stable.

(l) The system is a damped-driven harmonic oscillator, e.g. a forced mass-spring system with friction. The answer to parts (b) and (c) show clearly that the system is critically damped.

d) $\lambda_1 = \lambda_2 = -1 \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\boxed{x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \left[t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]}$$

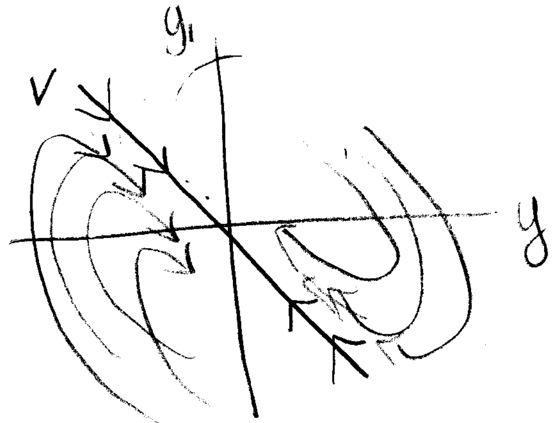
e) $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$r = -1,\ r = -1$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_2 e^{-t}}$$

f)



g) $y'' + 2y' + y = \cos t$

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t + -2A\sin t + 2B\cos t + A\cos t + B\sin t = \cos t$

$\cos t: -\cancel{A} + 2B + \cancel{A} = 1 \quad 2B = 1 \quad B = 1/2$

$\sin t: -\cancel{B} - 2A + \cancel{B} = 0 \quad -2A = 0 \quad A = 0$

$$\boxed{y_p = \tfrac{1}{2}\sin t}$$

$y_p' \ \tfrac{1}{2}\cos t$

$y_p'' = -\tfrac{1}{2}\sin t$

h) $y(0) = 0 \quad y'(0) = 1$

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_2 e^{-t} + \tfrac{1}{2}\sin t$

$y'(t) = -c_1 e^{-t} - c_2 t e^{-t} + \cancel{e^{-t} c_2} + \cancel{-c_2 e^{-t}} + \tfrac{1}{2}\cos t$

$= -c_1 e^{-t} - c_2 t e^{-t} + \tfrac{1}{2}\cos t$

$0 = c_1 + c_2$

$1 = -c_1 + \tfrac{1}{2}$

$c_1 = -\tfrac{1}{2}$

$c_2 = \tfrac{1}{2}$