Introduction to Partial Differential Equations

Homework 2

due March 8, 2017

1. Evans, p. 86 problem 4.

2. In class we have used Liouville's theorem to show that any bounded solution of the Poisson equation
$$-\Delta u = f$$
for $f \in C^2_{\mathrm{c}}(\mathbb{R}^n)$, $n \geq 3$, is given by the solution formula
$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy$$
up to an arbitrary additive constant. (Evans, p. 30, Theorem 8.)

   This statement does not hold in dimension $n = 2$ since solutions are generically unbounded. Use Liouville's theorem to conjecture and prove the corresponding theorem for $n = 2$.

3. Let U be open and bounded with a C^1 boundary. For every $v \in C^2(\bar{U})$, set
$$J[v] = \int_U \left(\tfrac{1}{2}\, |Dv|^2 - f\, v \right) dx - \int_{\partial U} g\, v\, dS\,.$$
Assume thoughout that $u \in C^2(\bar{U})$. Prove that the following two statements are equivalent.

   (i) u solves the so-called *Neumann problem*
$$\begin{aligned} -\Delta u &= f \qquad \text{in } U\,, \\ \nu \cdot Du &= g \qquad \text{on } \partial U\,. \end{aligned}$$

   (ii) u minimizes J, i.e.
$$J[u] \leq J[w]$$
for every $w \in C^2(\bar{U})$.

4. Evans, p. 87 problem 10.

5. Evans, p. 87 problem 11.