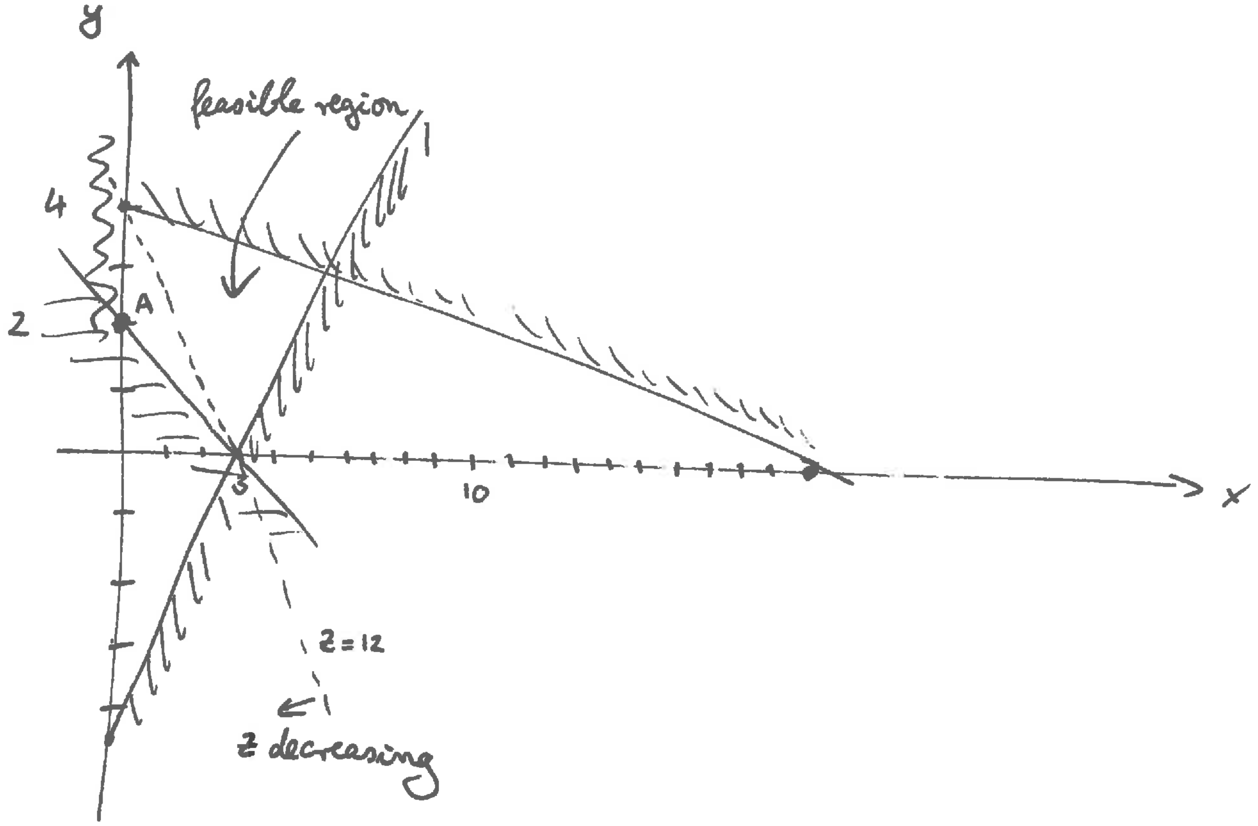
1. Use the graphical method to minimize

$$z = 4x + 3y$$

subject to

$$\begin{aligned} 2x + 3y &\geq 6, \\ 3x - 2y &\leq 9, \\ x + 5y &\leq 20, \\ x, y &\geq 0. \end{aligned}$$

(20)



By inspection, we can decrease the value of z by parallel translating the line z = const until it crosses point A = (0,2).

Then, $z = 4\cdot 0 + 3\cdot 2 = 6$.

2. The Steelworks have three machines each of which has t_i hours of spare operating time available. They can be used to make four products, each with possible profit of p_j Euros per unit sold. The production of one unit of product j requires w_{ij} hours of working time on machine i.

The production schedule for the week has been planned using the attached Pyomo program. Look at the program output to answer the following questions.

(a) Marketing determines that product P02 will still sell if the unit price is raised by 20 Euros. Give an estimate of the resulting increase in profits.

(b) On Tuesday, three hours before the end of the last shift, the cutting machine breaks down. The service engineer needs the rest of the day to fix the machine, but is certain that it will be back in service the next morning. Will this affect the planned weekly output?

(c) One of the workers would like to borrow the welding machine to mend his fender. His boss is inclined to grant the request, but insists that the cost in lost profits to the company must be covered by the borrower. How much should he charge if the machine is used for two hours?

(d) In an alternative scenario, the welding machine is out for 10 hours. Do you expect that the loss in profits is 5 times your answer to question (c)? What do you advise?

(5+5+5+5)

(a) Output A says that P02 is produced at 10 units p.w., so increasing its price will raise profits by at least $10 \cdot 20€ = 200€$. (Note: it could be more if the solution would jump to another vertex of the feasible region.)

(b) Output B says that the cutting machine has an upper slack of 4.5 h, which is more than the downtime of 3 h. ⇒ No change to production.

(c) Output C says that the shadow price for the welding machine is 22.73 €/h, so the price estimate based on marginal utility is 45.46 €.

(d) Data D means that downtime is almost 50% of available time, which is not a small change. The optimal solution is likely to jump to a different vertex of the feasible region. It is necessary to re-solve the LP with the new data.

(Remark: doing so yields that the loss in profit is indeed much bigger than 10 times the shadow price!)

In [1]:
```
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```
t = {'Cut': 38, 'Roll': 28, 'Weld': 21}  D
p = {'P01': 260, 'P02': 350, 'P03': 250, 'P04': 370}
w = {('Cut', 'P01'): 1.1,
     ('Cut', 'P02'): 2.5,
     ('Cut', 'P03'): 1.7,
     ('Cut', 'P04'): 2.6,
     ('Roll', 'P01'): 1.7,
     ('Roll', 'P02'): 2.1,
     ('Roll', 'P03'): 1.4,
     ('Roll', 'P04'): 2.4,
     ('Weld', 'P01'): 1.6,
     ('Weld', 'P02'): 1.3,
     ('Weld', 'P03'): 1.6,
     ('Weld', 'P04'): 0.8}

P = list(p.keys())
M = list(t.keys())

model = ConcreteModel()
model.x = Var(P, within=NonNegativeReals)

def profit_rule(model):
    return sum(p[j]*model.x[j] for j in P)
model.profit = Objective(rule=profit_rule, sense=maximize)

def capacity_rule(model, i):
    return sum(w[i,j]*model.x[j] for j in P) <= t[i]
model.capacity_constraint = Constraint(M, rule=capacity_rule)

model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
```

In [3]:
```
model.x.get_values()
```

Out[3]:
```
{'P01': 0.0, 'P02': 10.0, 'P03': 5.0, 'P04': 0.0}
```
A

In [4]:
```
model.profit.expr()
```

Out[4]:
```
4750.0
```

In [5]:
```
[(i,
  model.dual[model.capacity_constraint[i]],
  model.capacity_constraint[i].uslack())
 for i in M]
```

Out[5]:
```
[('Cut', 0.0, 4.5),  B
 ('Roll', 152.597402597403, 0.0),
 ('Weld', 22.7272727272728, 0.0)]
```
C

3. Prove that if there are feasible solutions for both primal and dual form of a linear programming problem, then both must have an optimal solution. (10)

The standard primal problem provides a lower bound to the dual objective function by weak duality. Since the dual feasible region is closed, it must have an optimal solution.

Vice versa, the dual provides an upper bound to the primal objective function, so an optimal value must exist (by closeness of the feasible region).

<u>Note</u>: To be mathematically rigorous, the appeal to closedness of the feasible region is required, otherwise an approximating sequence of near-optimal solutions may converge (or possess a converging subsequence) to a point outside of, but on the boundary of the feasible region.
Since this is a mathematical subtlety outside of the scope of this class, I gave full credit if the lower bound argument was stated in some form.

4. Andy, Bob, and Curt together with Doris, Erica, and Fiona visit a desert island. Each women is spending a certain fraction of her time with one of the men and none alone; likewise, each man is spending a certain fraction of his time with one of the women and none alone. The following table gives a measure of the happiness that ensues from each possible couple spending time together.

	Doris	**Erica**	**Fiona**
Andy	1	3	3
Bob	4	3	2
Curt	5	4	2

(a) Formulate a linear programming problem to maximize overall happiness on the island, assuming that happiness is additive and proportional to the fraction of time each couple spends together.
(You do not need to solve this problem.)

(b) Do you expect the result to suggest monogamous or polygamous relationships? Explain.

(10+5)

(a) Let x_{ij} denote the fraction of time that man i spends with woman j, and let c_{ij} denote the coefficients from the table.

The task is then to

$$\text{maximize} \quad z = \sum_{i,j} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_j x_{ij} = 1 \quad \text{for every } i \in \{A,B,C\}$$

$$\sum_i x_{ij} = 1 \quad \text{for every } j \in \{D,E,F\}$$

$$x_{ij} \geq 0$$

(b) This is an assignment problem (a special case of a transportation problem). As such, it satisfies the integrality property: Since the constraint coefficients are integer, the x_{ij} must be integer, too. Due to the constraints, this means that $x_{ij} \in \{0,1\}$. I.e., the resulting relationships are monogamous unless there is degeneracy so that other convex combinations are also possible.

5. Recall that the primal form of the standard activity analysis problem

$$\begin{aligned} &\text{maximize } \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to } A\boldsymbol{x} \le \boldsymbol{b}\,, \\ &\boldsymbol{x} \ge 0 \end{aligned} \tag{P}$$

has the symmetric dual

$$\begin{aligned} &\text{minimize } \boldsymbol{y}^T\boldsymbol{b} \\ &\text{subject to } \boldsymbol{y}^T A \ge \boldsymbol{c}^T\,, \\ &\boldsymbol{y} \ge 0\,. \end{aligned} \tag{D}$$

Here, A is an $n \times m$ matrix, $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^m$, and $\boldsymbol{y}, \boldsymbol{b} \in \mathbb{R}^n$.

Now a certain type of application leads to the linear programming problem

$$\begin{aligned} &\text{maximize } s \\ &\text{subject to } s \le B\boldsymbol{p}\,, \\ &p_1 + \cdots + p_\ell = 1\,, \\ &\boldsymbol{p} \ge 0\,, \end{aligned} \tag{P'}$$

where B is a given $k \times \ell$ matrix. There are $\ell + 1$ decision variables, namely $s \in \mathbb{R}$ and $\boldsymbol{p} \in \mathbb{R}^\ell$. Note that there is no sign constraint on s and there is one equality rather than inequality constraint.

(a) Write this problem in the standard form (P) by introducing an artificial non-negative variable and representing the equality by two inequalities. In particular, write out A, $\boldsymbol{b}$, and $\boldsymbol{c}$ for a given matrix B and state their dimensions.

(b) Show that problem (P') also has a symmetric dual formulation, namely

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } t \ge B^T\boldsymbol{q}\,, \\ &q_1 + \cdots + q_k = 1\,, \\ &\boldsymbol{q} \ge 0\,. \end{aligned} \tag{D'}$$

(5+10)

(a) Write $s = u - v$, $u \geq 0$, $v \geq 0$. Then problem becomes

maximize $u - v$

subject to $u - v - Bp \leq 0$

$p_1 + \cdots + p_\ell \leq 1$

$-p_1 + \cdots + (-p_\ell) \leq -1$

$p_1, \ldots, p_\ell, u, v \geq 0$

In vector notation:

$$x = \begin{pmatrix} p_1 \\ \vdots \\ p_\ell \\ u \\ v \end{pmatrix} \qquad c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

and
$$A = \left(\begin{array}{ccc|cc} & -B & & 1 & -1 \\ & & & \vdots & \vdots \\ & & & 1 & -1 \\ \hline 1 & \cdots & 1 & & \\ -1 & \cdots & -1 & & 0 \end{array}\right) \in \mathrm{Mat}\,\big((k+2)\times(\ell+2)\big)$$

so that the problem reads

$$\text{maximize } c^T x$$
$$\text{subject to } Ax \leq b, \quad x \geq 0.$$

(b) The dual problem, minimize $y^T b$

$$\text{subject to } y^T A \geq c^T, \quad y \geq 0$$

reads, setting $t = y_{k+1} - y_{k+2}$, $q^T = (y_1, \dots, y_k)$:

$$\text{minimize } t$$

subject to $-B^T q + y_{k+1} - y_{k+2} \geq 0 \Rightarrow t \geq B^T q$

and
$$\left.\begin{array}{l} q_1 + \dots + q_k \geq 1 \\ -q_1 + \dots + (-q_k) \geq -1 \end{array}\right\} \Rightarrow q_1 + \dots + q_k = 1$$

and $q \geq 0$.

6. A railway line is used for passenger and freight traffic in a single direction. The line is composed of 6 signal blocks. Each block can be used by only one train at a time, but at the start of each signal block there is a siding where another train can be kept waiting.

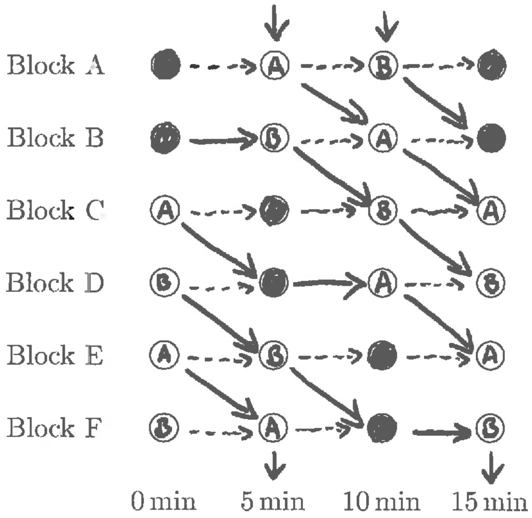
For simplicity, we analyze the problem in discrete 5-minute time slots. Trains enter a signal block exactly at the start of each slot.

A freight train takes exactly 5 minutes to traverse a signal block. Passenger trains travel twice as fast and must not be delayed. Thus, the block entered by a passenger train *as well as one block ahead* must be free of moving freight trains, but there could be a freight train kept waiting in one of the sidings.

This problem can be analyzed as a maximum flow problem for the movement of freight trains on a network whose nodes represent the signal blocks at the start of each time slot.

(a) Black dots occupied by passenger train

(b) All arrows indicate feasible movements.
Note: In a block occupied by a passenger train, no forward moving freight train is allowed!



Every 15 minutes, a passenger train enters the the first block A. The schedule must repeat every 15 minutes, so that the occupancy of blocks at $t = 0\,\text{min}$ and $t = 15\,\text{min}$, as well as the respective in- and outflows must coincide.

(a) Mark the signal blocks occupied by the passenger train in the network graph.

(b) Indicate with arcs the possible movement of freight trains in the network. Each node in the top layer can receive unrestricted inflow, each node at the bottom layer can have an outflow subject to the restrictions explained above.

(c) How many freight trains can enter the line during each 15 minute period?

(d) How much time do to the freight trains spend waiting on a siding?

(c) 2 are possible, their movement is indicated by solid arrows.

(5+5+5+5)

(d) 5 min for first freight train "A" and 10 min for second train "B".