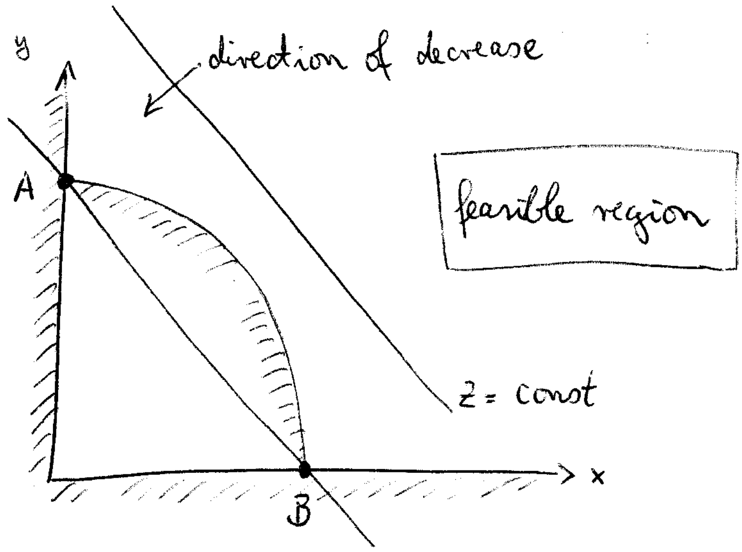
1. Use the graphical method to minimize

$$z = x + y$$

subject to

$$x^2 + y^2 \geq 1\,,$$
$$x, y \geq 0\,.$$

(20)



The equation $x^2+y^2=1$ describes a circle of radius 1 centered at the origin. Thus, the feasible region is the entire first quadrant outside of this circle.

The objective function is constant on lines of slope -1 and is decreasing toward the origin.

Thus, it take its minimum value at both points $A=(0,1)$ and $B=(1,0)$ where $z=1$.

2. Paula is using the Pyomo notebook displayed on the next page to optimize the shipment of containers, described by the decision variables x_{ij}, for her company.

   (a) Which problem is she solving? Describe the problem in words. You may complete the model scenario in any way consistent with the given information.

   (b) Which problem is she solving? State the problem in traditional mathematical notation.

   (c) A container must be shipped as a unit. Paula remembers the "integrality property" from her OR class and is surprised by the solution she is obtaining. Can you explain what is going on? How can she obtain an integer solution?

   (d) Give a *brief* interpretation of the program output shown in cells [4] and [5].

(5+5+5+5)

(a) It's a transportation problem. At ports P1, P2, P3 there are s_i containers to be transported to markets M1, M2, M3 with demand d_j. The cost of transportation from P_i to M_j is c_{ij}. Paula is minimizing the total cost of transportation.

(b)
$$\text{minimize} \quad \sum_{i,j=1}^{3} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} = s_i \qquad \text{for each } i=1,\dots,3$$

$$\sum_{i=1}^{3} x_{ij} = d_j \qquad \text{for each } j=1,\dots,3$$

$$x_{ij} \geq 0 \qquad \text{for } i,j=1,\dots,3 .$$

(c) Paula is using ipopt as a solver. A transportation problem with integer coefficients has an integer optimal solution (integrality property), but when the problem has multiple optimal solutions, Ipopt will typically return a solution which is not a vertex of the feasible region, thus a non-integer solution.

Paula should use a simplex-based solver such as GLPK instead.

Note: simply changing the code to

within = NonNegativeIntegers

will not work as ipopt is fundamentally unable to enforce integer constraints and will simply ignore the request.

(d) Cell [4] prints the demand duals, or shadow prices. They specify the marginal increase of transportation cost per unit increase of demand at the respective market.

Cell [5] prints the supply duals (shadow prices). Being negative, they specify the marginal decrease of transportation cost per unit increase of supply at the respective port.

The absolute value of the shadow price can be seen as the fair market price of one container under the assumption that the price only depends on the transportation cost and all other costs are zero.

In [1]:
```python
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("ipopt")
```

In [2]:
```python
P = ['P1', 'P2', 'P3']
M = ['M1', 'M2', 'M3']

s = {'P1':3, 'P2':5, 'P3':1}
d = {'M1':2, 'M2':2, 'M3':5}

c = {('P1','M1'):10, ('P1','M2'):5,  ('P1','M3'):15,
     ('P2','M1'):15, ('P2','M2'):10, ('P2','M3'):20,
     ('P3','M1'):5,  ('P3','M2'):5,  ('P3','M3'):15}

model = ConcreteModel()
model.x = Var(P, M, within=NonNegativeReals)

model.z = Objective(expr = sum(c[i,j]*model.x[i,j] for i in P for j in
M), sense=minimize)

def supply_rule (model, i):
    return sum(model.x[i,j] for j in M) <= s[i]
model.supply = Constraint(P, rule=supply_rule)

def demand_rule (model, j):
    return sum(model.x[i,j] for i in P) >= d[j]
model.demand = Constraint(M, rule=demand_rule)

model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
model.x.get_values()
```

Out[2]:
```
{('P1', 'M1'): 0.4634679251867636,
 ('P1', 'M2'): 0.8561172866483512,
 ('P1', 'M3'): 1.6804148178139067,
 ('P2', 'M1'): 0.5365320260507703,
 ('P2', 'M2'): 1.1438827029955108,
 ('P2', 'M3'): 3.3195851417491173,
 ('P3', 'M1'): 1.0000000288526543,
 ('P3', 'M2'): 0.0,
 ('P3', 'M3'): 0.0}
```

In [3]:
```python
model.z.expr()
```

Out[3]:
```
124.99999835228026
```

In [4]:
```python
for j in M:
    print(j, model.dual[model.demand[j]])
```

```
M1 15.043624766149037
M2 10.043624768629797
M3 20.043624770065662
```

In [5]:
```python
for i in P:
    print(i, model.dual[model.supply[i]])
```

```
P1 -5.043624771556779
P2 -0.043624770820542835
P3 -10.043624768654922
```

3. Use dynamic programming to solve the following *integer* nonlinear programming problem.

$$\text{Minimize } z = 3\,x_1 + 2\,x_2^2 + x_3^3$$
$$\text{subject to } x_1 + x_2 + x_3 = 4\,,$$
$$x_1, x_2, x_3 \geq 0\,.$$

(20)

Let $s_i = 4 - x_1 - \dots - x_i$ (the part of the equality constraint "used up" up to and including the i-th stage.

$f_3^*(s_2) = x_3^3 = s_2^3$ (there is no choice at stage 3 !)

At stage $i=2$: $f_2(s_1,s_2) = 2x_2^2 + f_3^*(s_2) = 2\,(s_1-s_2)^2 + f_3^*(s_2)$

s_1	$f_2(s_1,0)$	$f_2(s_1,1)$	$f_2(s_1,2)$	$f_2(s_1,3)$	$f_2(s_1,4)$	$f_2^*(s_1)$	x_2^*
0	0	—	—	—	—	0	0
1	$2\cdot 1^2+0^3$	$0+1^3$	—	—	—	1	0
2	$2\cdot 2^2+0^3$	$2\cdot 1^2+1^3$	$0+2^3$	—	—	3	1
3	$2\cdot 3^2+0^3$	$2\cdot 2^2+1^3$	$2\cdot 1^2+2^3$	$0+3^3$	—	9	2
4	$2\cdot 4^2+0^3$	$2\cdot 3^2+1^3$	$2\cdot 2^2+2^3$	$2\cdot 1^2+3^3$	$0+4^3$	16	2

At stage $i=1$: $f_1(4,s_1) = 3x_1 + f_2^*(s_1) = 3(4-s_1) + f_2^*(s_1)$

$f_1(4,0)$	$f_1(4,1)$	$f_1(4,2)$	$f_1(4,3)$	$f_1(4,4)$	f_1^*	x_1^*
$3\cdot 4+0$	$3\cdot 3+1$	$3\cdot 2+3$	$3\cdot 1+9$	$3\cdot 0+16$	9	2

Thus, $z^* = f_1^* = 9$ with $x_1^* = 2$, $x_2^* = 1$, $x_3^* = 1$

4. Customs authorities have busted a smuggling ring specialized in exclusive watches. Authorities are now selling off the seized merchandise for EUR 250 a piece. It is known that the true value of each hand-made watch is EUR 2000, but also that there are many cheap replicas in circulation with a market value of only EUR 40 a piece. The conditions of sale do not provide any warranty whether a watch is genuinely hand-made. Prior to purchase, only a visual inspection is allowed. A lay person will not be able to tell a genuinely hand-made watch from a replica.

   (a) The trade press has recently reported that 90% of all watches circulating on the black market are replicas. Given this information, would you choose to acquire watches at the quoted price?

   (b) An expert is offering her services to do a visual pre-screening of the watches. In the past, she has been correct 75% of the time. Compute the probability that she will classify one of the watches as genuinely hand-made.

   (c) Given that the expert classifies a watch as hand-made, what is the probability that the watch is truly hand-made?

   (d) What is the maximal fee per watch you should be willing to pay for her expert evaluation?

(5+5+5+5)

(a) We compute the expected value of the purchase using the prior probabilities:

$$\mathbb{E}[V] = 2000 \cdot \frac{1}{10} + 40 \cdot \frac{9}{10} = 200 + 36 = 236$$

as this is less than the sale price, the lay buyer will incur an expected loss, the advice is not to buy.

(b) $P(\text{test genuine}) = P(\text{test genuine} \mid \text{truly genuine})\, P(\text{truly genuine})$

$+ P(\text{test genuine} \mid \text{fake})\, P(\text{fake})$

$$= \frac{3}{4} \cdot \frac{1}{10} + \frac{1}{4} \cdot \frac{9}{10} = \frac{12}{4 \cdot 10} = \frac{3}{10}$$

(c) Bayes' rule:

$$P(\text{truly genuine} \mid \text{test genuine}) = \frac{P(\text{test genuine} \mid \text{truly genuine})\, P(\text{truly genuine})}{P(\text{test genuine})}$$

$$= \frac{\frac{3}{4} \cdot \frac{1}{10}}{\frac{3}{10}} = \frac{1}{4}$$

(d) If expert says "it's genuine", then

$$\mathbb{E}[V \mid \text{test genuine}] = 2000 \cdot \frac{1}{4} + 40 \cdot \frac{3}{4} = 530$$

so the decision is "buy".

If expert says "it's fake", then expected value is less than in part (a), clearly the decision is "do nothing".

Thus, the total expected value is

$$\mathbb{E}[V] = \frac{3}{10} \cdot (530 - 250) = \frac{3}{10} \cdot 280 = 84$$

$\Rightarrow$ The cost of the expertise should be less or equal to EUR 84 per watch.

Note: The valuation above assumes that you choose a watch (at random) and pass it to the expert for evaluation. If the setting is such that the expert can choose, from a large pool of watches, one that she classifies as genuine, then the expected payoff is EUR 530, so the value of information is EUR 280.

5. An airplane manufacturer is contracted to produce a small number of a particular type of airplane during the coming years. The manufacturer will need to decide each year whether to set up a production run with a fixed set-up cost of EUR 1 000 000 per run. During each production run, the manufacturer can make at most 6 airplanes. If an airplane is not delivered during the year it is produced, it will incur a holding cost of EUR 100 000 per year. The number of airplanes required are $r_1 = 1$, $r_2 = 6$, $r_3 = 2$, and $r_4 = 3$ during each of the years. (20)

Year 4: $f_4^*(S_3) = \begin{cases} 0 & \text{if } S_3 = 3 \\ K & \text{if } S_3 < 3 \end{cases}$

Note: S_3 = number of planes in storage at end of year 3

$K = 1$ (set-up cost in M€)

Year 3: $f_3(S_2, S_3) = \begin{cases} 0 & \text{if } S_3 = S_2 - 2 \\ K & \text{if } S_3 > S_2 - 2 \end{cases}$ (no production in year 3

$+ S_3 \cdot h + f_4^*(S_3)$ (h: holding cost of 0.1 M€)

Note that $S_2 = 0, 2, 5$; all other values create additional holding costs without reducing setup costs.

S_2	$f_3(S_2, 0)$	$f_3(S_2, 3)$	$f^*(S_2)$	p_3^*
0	2K	K + 3h	K+3h	5
2	K	K + 3h	K	0
5	-	3h	3h	0

Year 2: Note that we must produce in year 2!

$$f_2(s_1, s_2) = K + h s_2 + f_3^*(s_2)$$

The only values for s_1 which are not clearly suboptimal are $s_1 = 0, 2, 5$

s_1	$f_2(s_1,0)$	$f_2(s_1,2)$	$f_2(s_1,5)$	$f_2^*(s_1)$	P_2^*
0	$K+(K+3h)$	-	-	$2K+3h$	6
2	$K+(K+3h)$	$K+2h+K$	-	$2K+2h$	6
5	$K+(K+3h)$	$K+2h+K$	$K+5h+3h$	$K+8h$	6

Year 1: Again, we must produce in year 1, so

$$f_1(0, s_1) = K + s_1 h + f_2^*(s_1)$$

$f_1(0,0)$	$f_1(0,2)$	$f_1(0,5)$	f_1^*	P_1^*
$K + 2K + 3h$ $= 3K + 3h$ $= 3.3$ M€	$K+2h+(2K+2h)$ $= 3K + 4h$ $= 3.4$ M€	$K+5h+(K+8h)$ $= 2K + 13h$ $= 3.3$ M€	3.3 M€	1 or 6

So there are two optimal solutions:

(a) Produce 6 planes in year 1 and year 2 each,

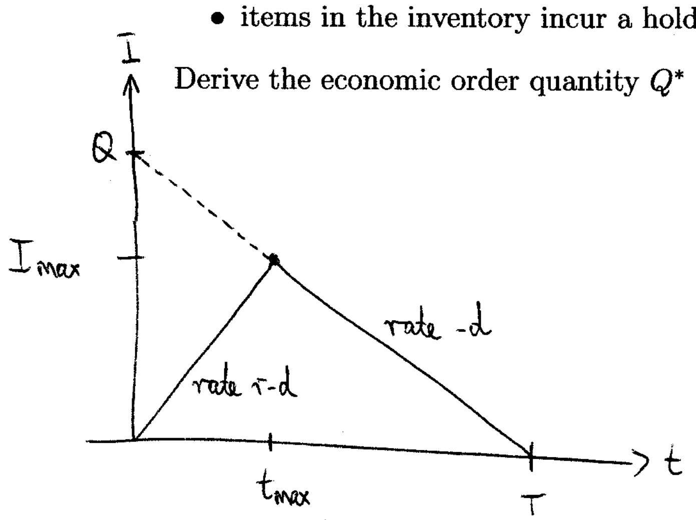
(b) Produce 1 plane in year 1, 6 in year 2, 5 in year 3.

6. Consider an inventory under the assumptions of the economic order quantity (EOQ) model with replacement at a constant rate, namely:

   - Constant demand rate of d units per time,
   - continuous review with no planned shortages,
   - an order with batch size Q is used to replenish the inventory at a constant rate $r > d$ units per time where replenishments and withdrawals can happen simultaneously,
   - each order incurs a setup cost K per batch,
   - items in the inventory incur a holding cost h per unit per time.

   Derive the economic order quantity Q^* as a function of r, d, h, and K.

   (20)



Note: Since $r > d$, the cycle time depends only on the slower withdrawal rate d:

$$T = \frac{Q}{d}$$

Further:
$$\left.\begin{aligned} I_{max} &= (r-d)\, t_{max} \\ Q &= r\, t_{max} \end{aligned}\right\} \Rightarrow I_{max} = Q\,\frac{r-d}{r}$$

Cost per cycle:
$$C_{cycle} = \underbrace{K}_{\text{order cost}} + \underbrace{\tfrac{1}{2} h\, I_{max}\, T}_{\text{holding cost}}$$

Cost per time:
$$C = \frac{C_{cycle}}{T} = \frac{Kd}{Q} + \frac{1}{2} h\, I_{max} = Kd\,Q^{-1} + \frac{1}{2} h \frac{r-d}{r} Q$$

$$\frac{dC}{dQ} = -Kd\,Q^{-2} + \frac{1}{2} h \frac{r-d}{r} \quad \Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2Kd}{h}\,\frac{r}{r-d}}}$$