# Introduction to Partial Differential Equations

Homework 7

due April 22, 2015

1. Evans, p. 163 problem 5

2. Let $L$ be convex with superlinear growth and suppose that $g$ is uniformly Lipshitz as defined in class (and in Evans, p. 124) via

$$\operatorname{Lip}(g) = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|} < \infty\,.$$

   Show that if, for fixed $x \in \mathbb{R}^n$ and $t > 0$,

$$u(x,t) = \inf_{y\in\mathbb{R}^n} \left\{ t\,L\left(\frac{x-y}{t}\right) + g(y) \right\},$$

   then the infimum in this expression actually is really a minimum.

3. (From Evans, p. 164, Question 13.) Let $u \in C(\mathbb{R}\times[0,T])$ for some $T > 0$ be an integral solution to the scalar conservation law

$$\begin{aligned} u_t + F(u)_x &= 0 \qquad \text{in } \mathbb{R}\times(0,T)\,, \\ u &= g \qquad \text{on } \mathbb{R}\times\{t=0\}\,. \end{aligned}$$

   Assume further that for any fixed $t \in [0,T]$, $u(\,\cdot\,,t)$ has compact support in $\mathbb{R}$, and that $F(0) = 0$. Show that

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} g(x)\,dx$$

   for every $t \in [0,T]$.

4. (From Evans, p. 164, Question 14.) Compute explicitly the unique entropy solution of

$$\begin{aligned} u_t + u\,u_x &= 0 \qquad \text{in } \mathbb{R}\times(0,\infty)\,, \\ u &= g \qquad \text{on } \mathbb{R}\times\{t=0\}\,. \end{aligned}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x \,. \end{cases}$$

Draw the characteristic curves in an $(x, t)$-plot, being sure to document all qualitative changes in the solution for $t > 0$.