Introduction to Partial Differential Equations

Homework 6

due April 8, 2015

1. Evans, p. 88 Problem 17.

2. For $h \in C^2(\mathbb{R}^3)$ with compact support and define, for $t > 0$,

$$u(x,t) = \fint_{\partial B(x,t)} t\, h(y)\, dS(y)\,.$$

   Show that

   (a) There exists $C > 0$ such that $|u(x,t)| \leq C/t$ for all $t > 0$.

   (b) $\lim_{t \to 0} u(x,t) = 0$.

   (c) $\lim_{t \to 0} u_t(x,t) = h(x)$.

   (d) u solves the wave equation

$$u_{tt} - \Delta u = 0$$

   on $\mathbb{R}^3 \times (0, \infty)$.

3. Evans, p. 163 Problem 3.

4. (This question refers to the remark in Evans, p. 114.) Consider the scalar conservation law in one spatial dimension

$$u_t + F'(u)\, u_x = 0 \qquad \text{in } \mathbb{R} \times (0,\infty)\,,$$
$$u = g \qquad \text{in } \mathbb{R} \times \{t = 0\}\,.$$

   (a) Show that the function u defined through the implicit equation

$$u = g(x - t\, F'(u))$$

   is a solution provided

$$1 + t\, g'(x - t\, F'(u))\, F''(u) \neq 0\,. \tag{*}$$

   (b) Suppose that $F(u) = \frac{1}{2}\, u^2$. When does solvability condition (*) fail? Is this only a failure of the method of characteristics, or does it also correspond to a failure of the solution to the PDE?