1. Solve the inhomogeneous transport equation

$$u_t + u_x + t\,e^{-x} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,,$$
$$u(x, 0) = e^{-x} \qquad \text{on } \mathbb{R}\,.$$

(10)

write $z(s) = u(x(s), t(s))$

$\Rightarrow z'(s) = x' u_x + t' u_t$

$\Rightarrow$ Need $x' = t' = 1$, so $x(s) = x_0 + s$, $t(s) = s$

$$\Rightarrow \underbrace{z(t)}_{= u(x,t)} - \underbrace{z(0)}_{= e^{-(x-t)}} = -\int_0^t s\, e^{-x(s)} ds = -e^{-(x-t)} \underbrace{\int_0^t s e^{-s} ds}$$

$$= -se^{-s}\Big|_{s=0}^{s=t} + \int_0^t e^{-s} ds$$

$$= -(s+1)e^{-s}\Big|_{s=0}^{s=t}$$

$$= 1 - (t+1)e^{-t}$$

$$\Rightarrow u(x,t) = e^{t-x} - e^{t-x} + (t+1)e^{-x}$$

$$= (t+1)\, e^{-x}$$

2. Suppose $u\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic with

$$\int_{\mathbb{R}^n} |u|\, dx < \infty\,.$$

Show that this implies $u = 0$. (5)

Mean-value formula:

$$u(x) = \fint_{B(x,r)} u(x)\, dx$$

$$\leq \frac{\int_{B(x,r)} |u|\, dx}{|B(0,r)|}$$

$$\leq \frac{\int_{\mathbb{R}^n} |u|\, dx}{|B(0,r)|} \xrightarrow[r\to\infty]{} 0$$

So $0$ is an upper bound for $u(x)$ for any $x \in \mathbb{R}^n$.

The same computation with $u$ replaced by $-u$ shows that $0$ is also a lower bound.

□

3. Suppose $u$ is a harmonic function on $\mathbb{R}^n$ with $n \geq 2$ such that

$$u(x) = x_1 + x_2 \qquad \text{on } \partial B(0,1).$$

(a) What are the minimum and maximum values of $u$ on $\overline{B(0,1)}$?

(b) Find $u(0)$.

(5+5)

(a) Clearly, the extreme values on $\partial B(0,1)$ are taken when $x_3, \dots, x_n = 0$. Moreover, by the arith - geometric mean inequality, the sum $x_1 + x_2$ is largest under the constraint $x_1^2 + x_2^2 = 1$ when $x_1 = x_2$; likewise for the smallest value.

Thus, maximum and minimum on the boundary are

$$u_{\text{extrema}} = \pm \frac{2}{\sqrt{2}} = \pm \sqrt{2}.$$

By the maximum principle for harmonic functions, these are already the extreme values on $\overline{B(0,1)}$.

(b) By the mean-value formula,

$$u(0) = \fint_{\partial B(0,1)} u(x)\, dS$$



By symmetry, this clearly evaluates to 0.

Alternative solution for (a):

Use the Minkowski inequality to estimate

$$\left(\frac{x_1+x_2}{2}\right)^2 \leq \frac{x_1^2+x_2^2}{2}$$

$$\leq \frac{x_1^2+x_2^2+\dots+x_n^2}{2}$$

$$= \frac{1}{2}$$

$\Rightarrow \quad x_1 + x_2 \leq \sqrt{2}$

where equality occurs when $x_1 = x_2 = \frac{1}{\sqrt{2}}$.

Similarly for the minimum.

4. Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $u \in C^2(\overline{U})$ be a solution to the *Helmholtz equation* with Neumann boundary conditions,

$$\begin{aligned} -\Delta u + u &= f && \text{in } U, \\ \nu \cdot Du &= g && \text{on } \partial U. \end{aligned}$$

(a) Show that $u$ is the unique such solution.

(b) What can you say about uniqueness of solutions to the Poisson equation $\Delta u = f$ in otherwise the same setting?

(5+5)

(a) Let $u, \tilde{u}$ be two solutions and set $w = u - \tilde{u}$.

Then $w$ satisfies

$$\begin{aligned} \Delta w + w &= 0 && \text{in } U \\ \nu \cdot Dw &= 0 && \text{on } \partial U \end{aligned}$$

$$\Rightarrow \quad 0 = \int_U w(-\Delta w + w)\, dx = -\int_{\partial U} w \, \nu \cdot Dw \, dS + \int_U |Dw|^2 + w^2 \, dx$$

$$\geq \int_U w^2 \, dx$$

$$\Rightarrow \quad w = 0 \text{ in } \overline{U} \quad \Rightarrow \quad u = \tilde{u}$$

(b) For the Poisson equation, we can only conclude that $Dw = 0$, so $w = \text{const}$ on each connected component of $\overline{U}$.

5. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$
$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}$$

is given by

$$u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t)\, g(y)\, dy, \qquad (*)$$

where, for $t > 0$,

$$\Phi(z, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|z|^2}{4t}}.$$

(a) Show that for every $g \in L^1(\mathbb{R})$ there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x, t)| \leq \frac{c}{\sqrt{t}}.$$

(b) Show that for every $g = h_x$ with $h \in L^1(\mathbb{R})$ there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x, t)| \leq \frac{c}{t}.$$

(c) Give a qualitative explanation for (b) vs. (a).

(5+5+5)

(a) From $(*)$: $|u(x,t)| \leq \int_{\mathbb{R}} \underbrace{\Phi(x-y,t)}_{\leq \frac{1}{\sqrt{4\pi t}}} |g(y)| dy \leq \frac{1}{\sqrt{t}} \frac{\|g\|_{L^1}}{2\sqrt{\pi}}$

(b) $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\, h_x(y)\, dy$

$$= \int_{\mathbb{R}} \underbrace{\Phi_x(x-y,t)}_{= \frac{1}{\sqrt{4\pi t}} \frac{-2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}}} h(y)\, dy \qquad (**)$$



p.t.o.

Thus, we must find the maximum of the function

$$\phi(z) = z e^{-\alpha z^2}$$

on $\mathbb{R}$. Necessary condition:

$$0 = \phi'(z) = e^{-\alpha z^2} - 2\alpha z^2 e^{-\alpha z^2}$$

$$\Rightarrow \quad 1 = 2\alpha z^2$$

$$\Rightarrow \quad z = \frac{1}{\sqrt{2\alpha}} \qquad \text{(positive root corresponds to maximum)}$$

Thus, $$\phi_{max} = \frac{1}{\sqrt{2\alpha}} e^{-\alpha \frac{1}{2\alpha}} = \frac{C_1}{\sqrt{\alpha}} \qquad (***)$$

Back into (**), this implies

$$|u(x,t)| \leq \frac{C_2}{t^{3/2}} \max_{z \in \mathbb{R}} z e^{-\frac{z^2}{4t}} \|h\|_{L^1}$$

$$\leq \frac{C_3}{t} \|h\|_{L^1} \qquad \left(\text{use } (***) \text{ with } \alpha = \frac{1}{4t}\right)$$

(c) If $g = h_x$ with $h \in L_1$, then $g$ will have a zero, which acts as a sink for the "concentration" $u$.

Note that in the usual interpretation of the heat equation, such initial data is unphysical, but it can arise if $u$ models the deviation from equilibrium.

6. Consider the equation[1]

$$u_t - u_{xxt} + 3\,u\,u_x = 2\,u_x\,u_{xx} + u\,u_{xxx} \qquad \text{in } \mathbb{R}\times(0,\infty)\,,$$
$$u = g \qquad \text{on } \mathbb{R}\times\{t=0\}\,.$$

Suppose that $u$ is a smooth solution such that $u(x,t) \to 0$ as $x \to \pm\infty$ with $u_x$ and $u_{xx}$ bounded for every fixed $t \geq 0$.

Show that

$$M(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx$$

is a constant of the motion. (10)

$$\frac{1}{2}\frac{dM}{dt} = \int_{\mathbb{R}} u u_t + u_x u_{xt} \, dx$$

$$= \int_{\mathbb{R}} u\,(u_t - u_{xxt})\, dx$$ (integr. by parts using decay conditions to eliminate boundary terms)

$$= \int_{\mathbb{R}} u\left(-3uu_x + 2u_x u_{xx} + u\, u_{xxx}\right) dx$$

$$= \underbrace{-\int_{\mathbb{R}} (u^3)_x \, dx}_{=0} + 2\int_{\mathbb{R}} u u_x u_{xx}\, dx + \underbrace{\int_{\mathbb{R}} u^2 u_{xxx}\, dx}_{= -2\int u u_x u_{xx}\, dx}$$

$= 0$ by fundamental theorem of calculus

$= -2\int u u_x u_{xx}\,dx$ also by I. b. p.

$$= 0\,.$$

[1]This equation is most commonly known as the Camassa–Holm equation, though its derivation goes back to Fokas and Fuchssteiner.