1. A bipartite graph with vertex sets V_1 and V_2 is *complete* if there is an edge between every vertex in V_1 and every vertex in V_2.

   Notation: The complete bipartite graph with $|V_1| = n$ and $|V_2| = m$ is called $K_{n,m}$. $K_{1,m}$, for any $m \in \mathbb{N}$, is called a *star*.

   (a) What is the number of edges of $K_{n,m}$?

   (b) Show that if a graph is complete bipartite and is a tree, then it is a star.

   (c) Show that $K_{3,3}$ cannot be embedded in the plane.

(5+5+5)

(a) $|E| = nm$ since for each of n vertices there is an edge to every one of m vertices.

(b) For a tree, $|V| - |E| = 1$

$\Rightarrow n + m - nm = 1$

$\Rightarrow (n-1)(m-1) = 0$

$\Rightarrow n = 1$ or $m = 1$

(c) Note: this is the "utility problem" from class!

Each face is bounded by at least 4 edges, each edge bounds two faces, so $4|F| \leq 2|E|$, or $|F| \leq \frac{9}{2}$

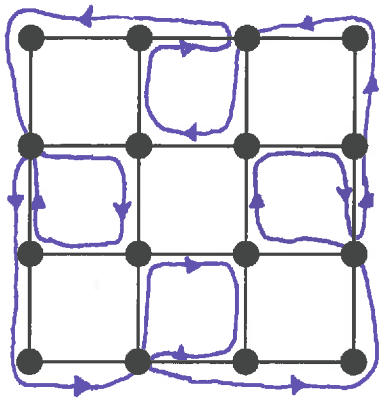
If the graph were planar, Euler's formula gives $\underbrace{|V|}_{6} - \underbrace{|E|}_{9} + |F| = 2$

$\Rightarrow |F| = 5$

Contradiction!

2. In the following graph, find the shortest closed path that traverses every edge at least once.

(a)



Your answer should include

(a) a correct solution;

(b) an argument that there is no shorter such path.

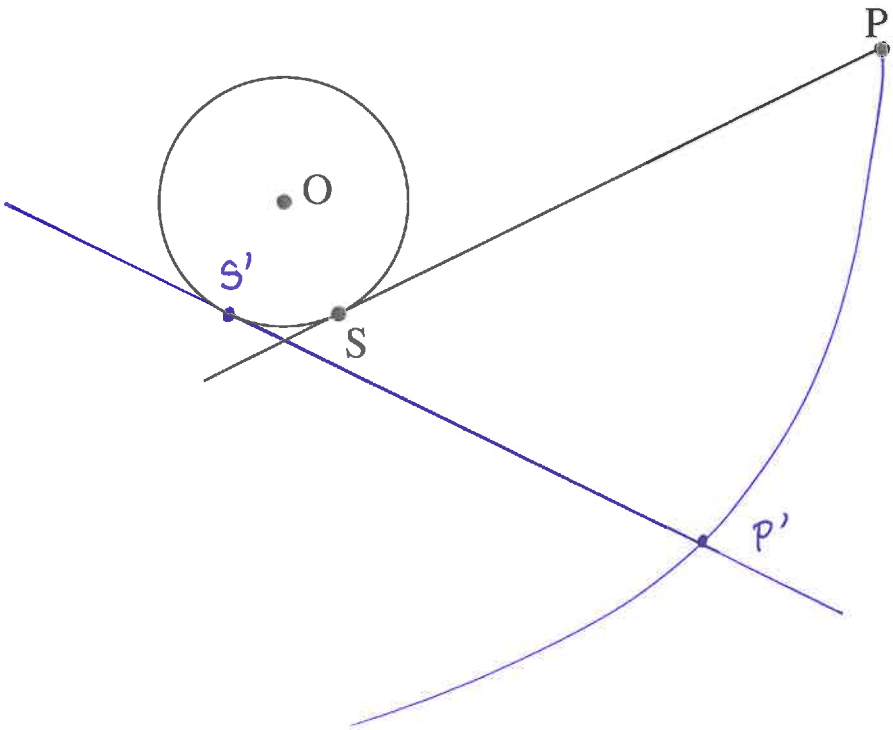
(5+5)

(b) If we consider edges visited twice as double-edges, the resulting non-simple graph must have vertices with even valency.

Since the original graph had 8 vertices with odd valency, we need at least 4 double edges.

Part (a) shows that this suffices.

3. Given a circle with center O and a point P outside of it, suggest a construction using only Euclidean transformations (translation, reflection, rotation) to find a point S on the circle where the line segment PS is tangent to the circle. (10)



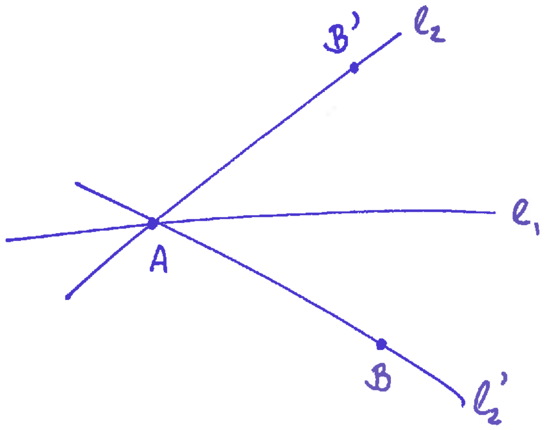
- Draw any tangent to the circle with point of tangency S'.
- Draw a circle about O through P.
- Let P' be a point of intersection of this circle with the tangent.
- Let Φ be the rotation about O that maps P' to P.
- Then $S = \Phi(S')$ is the sought-after point!

4. Given two lines ℓ_1 and ℓ_2, set $\ell_2' = R_{\ell_1}(\ell_2)$, i.e., ℓ_2' is the reflection of ℓ_2 about ℓ_1. Show that

$$R_{\ell_2} R_{\ell_1} = R_{\ell_1} R_{\ell_2'} .$$

(5)

Suppose ℓ_1 and ℓ_2 intersect in a point A:



We know both compositions are a rotation about A. To see that they coincide, it suffices to check the image of a single point $B \neq A$.

Let us take $B \in \ell_2'$

Then $\quad R_{\ell_2} R_{\ell_1} B = R_{\ell_2} B' \quad$ with $B' \in \ell_2$

$= B'$

$$R_{\ell_1} R_{\ell_2'} B = R_{\ell_1} B = B'$$

This proves the claim when $\ell_2 \not\parallel \ell_1$.

The argument is exactly parallel (sic) when $\ell_2 \parallel \ell_1$!

5. For each of the following, determine whether or not it is a group. If it is a group, give a full argument why; if it is not, state at at least one property that fails.

(a) The set of real $n \times n$ matrices under matrix multiplication;

(b) the orientation-preserving motions of the plane under composition;

(c) the orientation-reversing motions of the plane under composition.

(A motion or Euclidean transformations of the plane is called orientation-preserving if a clock-wise traversal of a cycle is mapped onto clock-wise traversal of its image; it is orientation-reversing otherwise.) (4+3+3)

(a) Not a group: Not all real $n \times n$ matrices have an inverse

(b) The motions of the plane are a group, so we only have to check that orientation-preserving motions are a subgroup.

- closedness under composition: the composition of two motions that preserve orientation must also preserve orientation, so O.K.
- closedness under inverse: if the mapping does not change orientation, the inverse must also preserve it, so O.K.

(c) Not a (sub-)group, as the identity is orientation preserving.

6. Let R_α denote the reflection about the line $x = \alpha$. Let G be the (symmetry) group generated by the unit translation along the x-axis and by R_0. Show that $R_\alpha \in G$ if and only if $2\alpha \in \mathbb{Z}$. (10)

$R_\alpha R_0 = T_{2\alpha}$ (as discussed in class)

Since all translations in G are of the form T_k, $k \in \mathbb{Z}$,

R_α is contained in G iff $2\alpha \in \mathbb{Z}$.