General Mathematics and Computational Science II

Final Exam

May 22, 2015

1. A bipartite graph with vertex sets V_1 and V_2 is *regular* if every vertex in V_1 has degree d_1 and every vertex in V_2 has degree d_2.

   Show that there exists a matching from V_1 into V_2 provided $d_1 \geq d_2 > 0$. (10)

2. Let H_A denote the point reflection about a point A with coordinates $\boldsymbol{a}$. Let $\Pi_{\boldsymbol{v}}$ denote the translation by a vector $\boldsymbol{v}$.

   (a) Show that $H_A \circ H_B = \Pi_{2(\boldsymbol{a}-\boldsymbol{b})}$.

   (b) Show that $H_A \circ H_B = H_B \circ H_C$ if and only if B is the midpoint of the line segment AC.

   (c) What can you say about the relative locations of A, B, C, and D in the case $H_A \circ H_B \circ H_C \circ H_D = \mathrm{Id}$?

   (5+5+5)

3. Let G_1 and G_2 be two finite Abelian groups, written additively. We define the direct sum $H = G_1 \oplus G_2$ as the set of ordered tuples $h = (g_1, g_2)$ with $g_1 \in G_1$ and $g_2 \in G_2$.

   (a) Define a +-operation on H that makes it an Abelian group.

   (b) Verify explicitly that with your definition you indeed obtain an Abelian group.

   (c) What is the order of H?

   (d) Let $\chi_i \colon G_i \to \mathbb{C}\backslash\{0\}$ be a character for $i = 1, 2$. Show that

   $$\psi(h) = \chi_1(g_1) \cdot \chi_2(g_2)$$

   defines a character on H.

   (Recall that $\chi_i \colon G_i \to \mathbb{C}\backslash\{0\}$ is a character if it is a group homomorphism into $\mathbb{C}\backslash\{0\}$, the group of nonzero complex numbers endowed with multiplication.)

   (e) Show that every character on H is of this form.

   (5+5+5+5+5)

4. Consider the linear programming problem

$$\text{maximize } z = 2\,x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\le 1\,, \\ x_1 - 2\,x_2 &\le 2\,, \\ x_1, x_2 &\ge 0\,. \end{aligned}$$

(a) Determine the solvability of this problem and find the solution if it exists, using the simplex method.

(b) Draw the feasible region and the level lines of the objective function; check for consistency with your answer in part (a).

(c) Write out the dual problem and determine its solvability. (No computation required.)

(10+5+5)

5. Show that the problems

$$A\boldsymbol{x} = \boldsymbol{b}\,, \qquad \boldsymbol{x} \ge \boldsymbol{0}$$

and

$$\boldsymbol{y}^T A \ge \boldsymbol{0}\,, \qquad \boldsymbol{y}^T \boldsymbol{b} < 0$$

cannot both have a solution. (10)

(In fact, *Farkas' lemma* states that excatly one of the two has a solution, but the complete proof is substantially more difficult.)

6. Recall that

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kjh}\, v_j$$

with $h = 2\pi/N$ denotes the discrete Fourier transform of the complex numbers $v_0, \dots, v_{N-1}$.

Define the discrete convolution of vectors $\boldsymbol{v} = (v_0, \dots, v_{N-1})$ and $\boldsymbol{w} = (w_0, \dots, w_{N-1})$ by

$$(\boldsymbol{v} \circledast \boldsymbol{w})_j = \frac{1}{N} \sum_{\ell=0}^{N-1} v_\ell\, w_{j-\ell}$$

where all indices are understood modulo N.

Show that

$$(\boldsymbol{v} \circledast \boldsymbol{w})\tilde{}_k = \tilde{v}_k\, \tilde{w}_k\,.$$

(10)

7. A 2π-periodic signal is sampled N times in the interval 2π. Suppose the signal contains only a fundamental frequency with wavenumber $k_0 = \frac{2}{5}N$ and its first harmonic at $k_1 = \frac{4}{5}N$.

   (a) Which frequencies does the trigonometric interpolant of the samples contain?

   (b) *Extra credit.* What does the reconstruction sound like? And what happens for different fundamental frequencies k_0?

(10+*10*)

Hint: Recall the aliasing formula

$$\tilde{u}_k = \sum_{m \in \mathbb{Z}} \hat{u}_{k+mN}$$

for $k = -N/2, \ldots, N/2 - 1$.