General Mathematics and CPS II

Exercise 22

May 8, 2015

1. Compute the discrete Fourier transform of the vector $(v_0, \ldots, v_{N-1})$ with N even and

$$v_j = \begin{cases} 1 & \text{for } j < N/2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   Note: The case $k = 0$ is special, but if you think of your expression for $\hat{v}_k$ as being defined for arbitrary $k \in \mathbb{R}$, you should find that $\lim_{k\to 0} \hat{v}_k = \hat{v}_0$.

2. Solve the difference equation

$$t_n = 2\,t_{n-1} + 2^{n+1}\,,$$
$$t_1 = 0\,.$$

3. The first question on the previous exercise sheet suggests a way to differentiate a periodic function $u\colon [0, 2\pi] \to \mathbb{R}$ numerically:

   - Sample u on the equidistant grid $x_j = jh$ with $h = 2\pi/N$.
   - Compute the discrete Fourier coefficients $\tilde{u}_k$ of the vector of samples.
   - Set $\tilde{w}_k = \mathrm{i}k\,\tilde{u}_k$.
   - Finally, take the inverse DFT to obtain an approximation for u' sampled on the grid.

   The approximation error for the derivative can be estimated in much the same way as the error estimate for trigonometric interpolation in Section 3 of the handout. Again, the reconstruction formula must use the symmetrized wave number range $k = -N/2, \ldots, N/2 - 1$.

   A practical difficulty arises from the convention used by virtually all standard FFT software libraries in which $j = 0, \ldots, N-1$ and $k = 0, \ldots, N-1$. In particular, the input and output data arrays are arranged in precisely this order. With which factor, then, must the ith element, with respect to the order imposed by the software, of the vector $(\tilde{u}_0, \ldots, \tilde{u}_{N-1})$ be multiplied so that differentiation is computed correctly?