General Mathematics and CPS II

Exercise 18

April 17, 2015

1. Use the simplex method on the following linear programming problem, applying the method from class to find the initial set of basic variables.

   Minimize
   $$z = -x_1 - x_2 - x_3$$
   subject to
   $$\begin{aligned} x_1 + 2\,x_2 + x_3 &= 4\,, \\ 2\,x_1 + x_2 + 2\,x_3 &= 10\,, \\ \boldsymbol{x} &\geq 0\,. \end{aligned}$$

2. The *primal* form of a linear programming problem is
   $$\begin{gathered} \text{minimize } \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\,. \end{gathered} \tag{P}$$
   The corresponding dual problem reads
   $$\begin{gathered} \text{maximize } \boldsymbol{y}^T\boldsymbol{b} \\ \text{subject to } \boldsymbol{y}^T A \leq \boldsymbol{c}^T\,. \end{gathered} \tag{D}$$
   Here, A is an $m \times n$ matrix, $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{y}, \boldsymbol{b} \in \mathbb{R}^m$.

   Show that if $\boldsymbol{x}$ solves (P) and $\boldsymbol{y}$ solves (D), then
   $$\boldsymbol{y}^T\boldsymbol{b} \leq \boldsymbol{c}^T\boldsymbol{x}\,.$$
   Conclude that the primal problem does not have a finite minimum if and only if the feasible region of the dual problem is empty.

3. In the notation of the previous question, show that if $\boldsymbol{x}$ is feasible for problem (P) and $\boldsymbol{y}$ is feasible for problem (D), and if furthermore
   $$\boldsymbol{y}^T\boldsymbol{b} = \boldsymbol{c}^T\boldsymbol{x}\,,$$
   then $\boldsymbol{x}$ solves (P) and $\boldsymbol{y}$ solves (D).