1. Show that $u \in W^{1,\infty}(\mathbb{R}^n)$ if and only if $u \colon \mathbb{R}^n \to \mathbb{R}$ is bounded and Lipshitz continuous. (10+10)

We follow the proof of Evans, p. 279, Theorem 4, noting that the assumtion U bounded is not really required in the argument.

"$\Rightarrow$": Let $u \in W^{1,\infty}(\mathbb{R}^n)$, set $u_\varepsilon = \eta_\varepsilon * u$ with η_ε the standard mollifier. Then

$$u_\varepsilon(x) - u_\varepsilon(y) = \int_0^1 \frac{d}{dt} u_\varepsilon(y + t(x-y))\, dt$$
$$= \int_0^1 (x-y) \cdot Du_\varepsilon(y + t(x-y))\, dt$$

$$\Rightarrow |u_\varepsilon(x) - u_\varepsilon(y)| \leq \|Du_\varepsilon\|_{L^\infty} |x-y| \leq \|Du\|_{L^\infty} |x-y|$$

It is known that $u_\varepsilon \to u$ pointwise a.e., so pick points x, y for which convergence holds. Thus, u is a.e. equal to a Lipshitz function, boundeness holds by assumption.

"$\Leftarrow$": Let, as in Evans, $D^{-h} u = \left(\frac{u(x - he_1) - u(x)}{-h}, \dots, \frac{u(x - he_n) - u(x)}{-h} \right)$

Suppose u is bounded and Lipshitz. Then

$$\|D^{-h} u\|_{L^\infty} \leq \mathrm{Lip}(u).$$

Thus, by Banach-Alaoglu, $\{D^{-h}u\}$ is rel. compact in L^∞ and we can extract a subsequence, still indexed by h, s.t.

$$D^{-h}u \overset{*}{\rightharpoonup} v \qquad \text{in } L^\infty(\mathbb{R}^n)$$

for some $v \in L^\infty(\mathbb{R}^n)$. In particular, for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u D\phi \, dx = \lim_{h\to 0} \underbrace{\int_{\mathbb{R}^n} u D^h\phi \, dx}_{= -\int_{\mathbb{R}^n} D^{-h}u \, \phi \, dx} \xrightarrow[h\to 0]{} -\int_{\mathbb{R}^n} v \phi \, dx$$

So $Du = v$ in the sense of weak derivatives and $u \in W^{1,\infty}$ because $u, v \in L^\infty$.

2. Let $U \subset \mathbb{R}^n$ be open and bounded with C^2 boundary. Consider the bi-harmonic equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } U, \\ u &= 0 \quad \text{on } \partial U, \\ \nu \cdot Du &= 0 \quad \text{on } \partial U. \end{aligned}$$

(a) Define a notion of weak solution for the bi-harmonic equation. Your answer should clearly state and define all the function spaces involved.

(b) Prove the existence of a unique weak solution according to your definition.
Hint: It may be useful to refer to the regularity theorem for uniformly elliptic second order problems, which says that a weak solution $\nu \in H_0^1(U)$ to

$$Lu = g \quad \text{in } U$$

satisfies

$$\|u\|_{H^2(U)} \leq c \left(\|u\|_{L^2(U)} + \|g\|_{L^2(U)} \right)$$

for some constant c independent of g.

(10+10)

(a) For sufficiently smooth u, v, Green's identity gives

$$\int_U \Delta^2 u \, v \, dx = \int_{\partial U} \nu \cdot D\Delta u \, v \, dS - \int_{\partial U} \Delta u \, \nu \cdot Dv \, dS + \int_U \Delta u \, \Delta v \, dx$$

So the boundary terms vanish provided v satisfies the boundary conditions of the homogeneous Dirichlet biharmonic problem. This motivates to seek weak solutions $u \in H_0^2(U)$ s.t.

$$\int_U \Delta u \, \Delta v \, dx = \int_U f v \, dx \qquad \forall \, v \in H_0^2(U)$$

where we require that $f \in L^2(U)$.

Note: Since $u=0$ on ∂U, tangential derivatives vanish trivially. So the requirement $u \in H_0^2(U)$ is not stronger than $u=0$ and $\nu \cdot Du = 0$ on ∂U in the trace sense!

(b) Use Lax-Milgram on $B(u,v) = \int_U \Delta u \, \Delta v \, dx$.

Continuity on $H_0^2 \times H_0^2$ is obvious. For coercivity:

If $u \in C_0^\infty(U)$,

$$\int_U (\Delta u)^2 \, dx = -\int_U D\Delta u \cdot Du \, dx = \int_U \underbrace{D^2 u : D^2 u}_{\equiv |D^2 u|^2} \, dx$$

This identity extends to $u \in H_0^2(U)$ by density.

Further, by the Poincaré inequality, applied twice,

$$\|D^2 u\|_{L^2}^2 \geq C_1 \|Du\|_{L^2}^2 \geq C_2 \|u\|_{L^2}^2 .$$

So, altogether, $\exists \, \alpha > 0$ s.t.

$$B(u,u) = \int_U (\Delta u)^2 \, dx \geq \alpha \|u\|_{H^2}^2 .$$

3. Let

$$Lu = -\sum_{i,j=1}^{n} D_i(a^{ij}(x)\, D_j u)$$

be a uniformly elliptic symmetric second order operator with bounded coefficients. Suppose $u \in H^1_{loc}(\mathbb{R}^n)$ satisfies

$$Lu = 0$$

in the sense of weak derivatives. Show that $u \in L^2(\mathbb{R}^n)$ implies that u is a constant. (10)

For $v \in C_0^\infty(\mathbb{R}^n)$, by the definition of the weak derivative,

$$0 = \int_{\mathbb{R}^n} v\, Lu\, dx = -\int_{\mathbb{R}^n} Dv^T A\, Du\, dx \quad \text{with } A = (a_{ij})$$

Set $u_\varepsilon = \eta_\varepsilon * u$, η_ε the standard mollifier, and

$$\zeta_R(x) = \varphi_R(|x|) \quad \text{with}$$

$$\varphi_R(r) = \begin{cases} 0 & \text{for } r > R \\ \frac{1}{4}(R-r)^2 & \text{for } r \in [R-1, R] \\ 1 & \text{for } r \in [0, \frac{R}{2}] \end{cases}$$

and φ_R smooth and monotonic otherwise with $|\varphi_R'| \leq 1$ for R large enough.

Now let $v = \zeta_R u_\varepsilon$ and

$$0 = \int_{\mathbb{R}^n} Du^T A\, D(\zeta_R u_\varepsilon)\, dx = \int_{\mathbb{R}^n} Du^T A\, Du_\varepsilon\, \zeta_R\, dx + \int_{\mathbb{R}^n} Du^T A\, u_\varepsilon\, D\zeta_R\, dx$$

(Strictly speaking v is only C_0^1, so we'd need another mollification, but it's clear that this is trivially implemented.)

Letting $\varepsilon \to 0$ gives

$$\int_{\mathbb{R}^n} \underbrace{Du^T A Du}_{\substack{\geq \theta |Du|^2 \\ \text{by ellipticity}}} \zeta_R \, dx = \underbrace{- \int_{\mathbb{R}^n} Du^T A u \, D\zeta_R \, dx}$$

$$\leq \int_{\mathbb{R}^n} |Du| \, |A| \, |u| \, \underbrace{|D\zeta_R|}_{\leq |\varphi_R'(|x|)|} \, dx$$

$$\leq \sup_{x \in \mathbb{R}^n} |A(x)| \int_{B_R \setminus B_{\frac{R}{2}}} \zeta_R^{\frac{1}{2}} |Du| \, \zeta_R^{-\frac{1}{2}} |\varphi_R'| \, |u| \, dx$$

Now note that for $r \in [R-1, R]$, by direct calculation,

$$|\varphi_R'| \leq \varphi_R^{\frac{1}{2}}$$

and the same holds true, possibly with a prefactor, on $[\frac{R}{2}, R-1]$.
So, applying the Cauchy-Schwarz inequality, we obtain altogether

$$\int_{\mathbb{R}^n} |Du|^2 \zeta_R \, dx \leq c \left(\int_{\mathbb{R}^n} |Du|^2 \zeta_R \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus B_{\frac{R}{2}}} |u|^2 \, dx \right)^{\frac{1}{2}}$$

$$\Rightarrow \int_{\mathbb{R}^n} |Du|^2 \zeta_R \, dx \leq c^2 \int_{\mathbb{R}^n \setminus B_{\frac{R}{2}}} |u|^2 \, dx$$

When $u \in L^2(\mathbb{R}^n)$, the RHS goes to zero as $R \to \infty$, so $|Du| = 0$ a.e.
We conclude that u equals a constant function (zero, in fact) a.e.

4. Let $U \in \mathbb{R}^n$ be open and bounded with smooth boundary, and $T > 0$. Prove that there is at most one smooth solution of the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u = u^2 \qquad & \text{in } U_T\,, \\ \nu \cdot Du = 0 \qquad & \text{on } \partial U \times [0,T]\,, \\ u = g \qquad & \text{on } U \times \{t = 0\}\,. \end{aligned}$$

(10)

Let u_1, u_2 be two solutions, set $v = u_1 - u_2$.

$$\Rightarrow \quad v_t - \Delta v = u_1^2 - u_2^2 = (u_1 - u_2)(u_1 + u_2) = v\,(u_1 + u_2)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_U v^2\,dx - \underbrace{\int_U v\,\Delta v\,dx}_{} = \int_U (u_1+u_2)\,v^2\,dx$$

$$= \int_{\partial U} v\,\underbrace{\nu \cdot Dv}_{\equiv 0}\,dS - \int_U |Dv|^2\,dx$$

$$\Rightarrow \frac{d}{dt}\int_U v^2\,dx \leq \underbrace{2\left(\|u_1\|_{L^\infty([0,T]\times U)} + \|u_2\|_{L^\infty([0,T]\times U)}\right)}_{=:\,c} \int_U v^2\,dx$$

$$\Rightarrow \quad \|v(t)\|_{L^2(U)} \leq \|v(0)\|_{L^2(U)}\; e^{ct}$$

$\Rightarrow \quad v(t) = 0$ provided it is initially so.

$\square$

5. Let $U \in \mathbb{R}^n$ be open and bounded with smooth boundary. Suppose $u \in W^{1,\infty}(U, \mathbb{R}^n)$ with $\nu \cdot u = 0$ on ∂U and assume that $\theta = \theta(x,t)$ is a smooth solution to

$$\partial_t \theta + u \cdot D\theta = 0 . \tag{*}$$

(a) Show that there exists a constant c such that

$$\|\theta(t)\|_{L^p}^p \le e^{ct} \|\theta(0)\|_{L^p}^p$$

for every $2 \le p < \infty$ and $0 \le t < \infty$.

(b) Conclude that

$$\|\theta(t)\|_{L^\infty} \le \|\theta(0)\|_{L^\infty} .$$

Hint: You may use that

$$\|\theta\|_{L^\infty} = \lim_{p\to\infty} \|\theta\|_{L^p} .$$

(c) For every fixed $a \in U$ consider the autonomous ordinary differential equation

$$\frac{d\phi(a,t)}{dt} = u(\phi(a,t)) .$$

Show that $\theta(x,t)$, implicitly defined via

$$\theta(\phi(a,t),t) = \theta(a,0)$$

solves (*). You may assume sufficient smoothness of all objects involved, and that $\phi(a,t) \in U$.

(d) Do you see a connection between the above and the result from Question 1? Explain.

(5+5+5+5)

(a) Multiply (*) with θ^{p-1} and integrate over U:

$$\int_U \theta_t \theta^{p-1}\, dx + \int_U \theta^{p-1}\, u \cdot D\theta \, dx = 0$$

$$\Rightarrow \frac{1}{p}\frac{d}{dt}\int_U \theta^p \, dx + \frac{1}{p}\underbrace{\int_U u \cdot D\theta^p \, dx}_{= \int_{\partial U} \underbrace{\nu \cdot u}_{=0}\, \theta^p \, dS - \int_U D\cdot u \, \theta^p \, dx} = 0$$

$$\Rightarrow \frac{d}{dt}\int_U \theta^p \, dx \leq \underbrace{\|D\cdot u\|_{L^\infty}}_{=:c} \int_U \theta^p \, dx$$

The claim now follows by integration in time.

(b) $$\|\theta(t)\|_{L^p} \leq \underbrace{e^{ct/p}}_{\to 1 \text{ as } p\to\infty} \|\theta(0)\|_{L^\infty}$$

The claim thus follows by letting $p\to\infty$ for every t fixed.

(c) Just check the proposed solution by differentiation:

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t}(\phi(a,t),t) + D\theta(\phi(a,t),t)\underbrace{\frac{d\phi}{dt}(a,t)}_{\equiv u(\phi(a,t),t)} = 0$$

So under the assumption that ϕ is surjective, we can remove the common argument to find

$$\frac{\partial\theta}{\partial t} + u\cdot D\theta = 0 .$$

(d) (**) shows that the solution at time $t>0$ is simply a rearrangement of the solution at $t=0$. So the L^∞ is preserved (see (b) above) and the L^p norm is preserved up to a Jacobian factor due to the change-of-variable formula applied to the L^p-integral.