Partial Differential Equations

Homework 6

due May 15, 2014

1. Evans, p. 426, Problem 6

2. Assume $W \subset \mathbb{R}^n$ is open, $u\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a smooth time-dependent vector field with flow ϕ_t, i.e. $\partial_t \phi_t(a) = u(\phi_t(a), t)$, and $W_t = \phi_t(W)$.

   Prove that

   (a) $\displaystyle \frac{d}{dt} \int_{W_t} \rho\, f \, dx = \int_{W_t} \rho\, (\partial_t + u \cdot \nabla) f \, dx$, where $\partial_t \rho + \nabla \cdot (\rho u) = 0$, and

   (b) $\displaystyle \frac{d}{dt} \int_{W_t} f \, dx = \int_{W_t} \partial_t f \, dx + \int_{\partial W_t} f \, \nu \cdot u \, dS.$

3. *Setting:* Recall the construction of the function spaces for solutions of the Navier–Stokes equations, where for $U \in \mathbb{R}^n$ open, bounded, with C^2 boundary,

$$\mathcal{V} = \{u \in C_c^\infty(U, \mathbb{R}^n) \colon \nabla \cdot u = 0\} \,,$$

   H is the closure of $\mathcal{V}$ in L^2, and V is the closure of $\mathcal{V}$ in H^1. As is shown in Constantin & Foias, for example,

$$H = \{u \in L^2(U, \mathbb{R}^n) \colon \nabla \cdot u = 0 \text{ and } \gamma(u) = 0\}$$

   where the divergence understood in the sense of weak derivatives and where γ is the normal trace operator satisfying $\gamma(u) = \nu \cdot u$ if $u \in C^\infty(\overline{U})$ which is a bounded operator from

$$E(U) = \{u \in L^2(U, \mathbb{R}^n) \colon \nabla \cdot u \in L^2(U, \mathbb{R})\}$$

   to $H^{-1/2}(U)$. It is further known that

$$V = \{u \in H_0^1(U, \mathbb{R}^n) \colon \nabla \cdot u = 0\} \,.$$

   Recall further that the *Stokes operator* $A\colon \mathcal{D}(A) \to H$ is defined by $A = -P\Delta$ where P is the L^2-orthogonal projector onto H and $\mathcal{D}(A) = H^2(U, \mathbb{R}^n) \cap V$. It is a positive self-adjoint operator with compact inverse. Hence, there exists a complete L^2-orthonormal sequence $\{w_j\}$ of eigenfuctions of A with corresponding eigenvalues λ_j, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ and $w_j \in \mathcal{D}(A)$ for $j \in \mathbb{N}$.

Thus, if

$$u = \sum_{j=1}^{\infty} u_j \, w_j \,,$$

we can define A^α for any $\alpha \in \mathbb{R}$ by

$$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha \, u_j \, w_j \,,$$

$$\langle u, v \rangle_{\mathcal{D}(A^\alpha)} = \sum_{j=1}^{\infty} \lambda^{2\alpha} \, u_j \, v_j \quad \text{when } u = \sum_{j=1}^{\infty} u_j \, w_j \text{ and } v = \sum_{j=1}^{\infty} u_j \, v_j$$

with domain

$$\mathcal{D}(A^\alpha) = \{ u \in H \colon \|u\|^2_{\mathcal{D}(A^\alpha)} = \langle u, u \rangle_{\mathcal{D}(A^\alpha)} < \infty \} \,.$$

Then, in particular, $V = D(A^{1/2})$ and, writing V^* for the L^2-dual of V, $V^* = \mathcal{D}(A^{-1/2})$ so that

$$\|v\|_{V^*} = \|A^{-1/2} v\|_{L^2} \quad \text{and} \quad \langle v, w \rangle_{V^*,V} = \langle A^{-1/2} v, A^{1/2} w \rangle_{L^2} \,.$$

Problem: In the setting above, let $u \in V$. Prove that, for $n \leq 3$, there exists a constant c such that

$$\|u \cdot \nabla u\|_{V^*} \leq c \, \|u\|_V^{\frac{n}{2}} \; \|u\|_H^{2-\frac{n}{2}} \,.$$

Hint: You will need the Sobolev inequality in the form given as equation (14) in the proof of Theorem 5.6.1 in Evans (you can also quote the result from other sources where the exponents are more explicit).

4. Suppose that

$$\begin{aligned} u_m &\rightharpoonup 0 \qquad \text{weakly in } L^2((0,T);V) \,, \\ u_m' &\rightharpoonup 0 \qquad \text{weakly in } L^p((0,T);V^*) \,. \end{aligned}$$

for some $p > 1$.

(a) Prove that

$$u_m \to 0 \qquad \text{in } C([0,T];V^*) \,.$$

(b) Then conclude that

$$u_m \to 0 \qquad \text{in } L^2((0,T);H) \,.$$

Hint: For part (a), use the fundamental theorem of calculus, estimate in the V^*-norm, and employ an argument similar to the one given in part (1) of the proof of Evans, Theorem 5.9.4 on p. 287.

You will also need to refer to the compactness of the embedding $V \subset H$.