Partial Differential Equations

Homework 5

due April 22, 2014

Let $U \subset \mathbb{R}^n$ be open, bounded, and with C^2 boundary, and set $U_T = U \times (0, T]$. Consider the semilinear parabolic equation

$$u' - \Delta u = f(u) \qquad \text{in } U_T\,,$$
$$u = 0 \qquad \text{on } \partial U \times [0, T]\,,$$
$$u = g \qquad \text{on } U \times \{t = 0\}\,,$$

where $u \colon \overline{U}_T \to \mathbb{R}$, $g \in L^2(U)$, and, for simplicity, assume that

$$f(u) = u^{p-1}$$

for some $p \geq 2$.

Modify the discussion of weak solutions for linear parabolic equations from Evans, Section 7.1 to the case above. Write out the argument completely, where you can be brief on those parts that need no modification and should be careful and detailed when considering the effect of the nonlinearity.

Some things to consider:

- You need to carefully define a notion of weak solution so that you can make sense of the nonlinear right hand side. In particular, you might need to require the solution to lie in certain space-time L^p spaces.
- In general, you cannot hope to get a solution for arbitrary T. Expect your *a priori* estimates on the sequence of approximate solutions to blow up in a finite time which depends on the initial data g. What you still need to show is that the blow up time is independent of the approximation parameter m.
- You should expect to find a restriction on p which makes the estimates possible.
- A most important tool is the Gagliardo–Nirenberg–Sobolev inequality. In the form given by Evans, Section 5.6, Theorem 1, you will need to use it with the L^2 norm on Du on the right hand side, and you will need to combine it with a Hölder inequality to get an estimate of the form

  $$\|u\|_{L^q} \leq C\, \|Du\|_{L^2}^{\theta}\, \|u\|_{L^2}^{1-\theta}\,.$$

  What is the relation between n, q, and θ? Does this estimate hold for $u \in H_0^1$?

- Note that if a sequence is weakly converging in L^p and strongly converging in L^q for, then it will converge strongly in every intermediate space L^r with $q \leq r < p$. (Why?) You will need to use this for spaces L^p in space and time when passing to the limit in the weak formulation.