Partial Differential Equations

Homework 3

due March 18, 2014

1. Prove the following version of the Poincaré inequality: Let $U \subset \mathbb{R}^n$ be open, bounded, with smooth boundary. Then there exists a constant c such that for every $u \in H_0^1(U)$,

$$\|u\|_{L^2(U)} \leq c \, \|Du\|_{L^2(U)} \, .$$

2. Let H be a Hilbert space and $A \colon H \to H$ a bounded linear operator. Show that

$$\overline{\operatorname{Range} A} = (\operatorname{Ker} A^*)^\perp \, .$$

3. Let H be a Hilbert space, and suppose $x_k \rightharpoonup x$ weakly in H. Show that

$$\|x\| \leq \liminf_{k \to \infty} \|x_k\| \, .$$

4. Study the proof of Theorem 6 on pp. 306–307 of Evans. Show that the Theorem actually holds true in the following stronger version.

   If $\lambda \notin \Sigma$, there exists a constant C such that

$$\|u\|_{L^2(U)} \leq C \, \|f\|_{H^{-1}(U)}$$

   whenever $f \in H^{-1}(U)$ and $u \in H_0^1(U)$ is the unique weak solution of

$$\begin{aligned} Lu &= \lambda h + f \qquad \text{in } U \, , \\ u &= 0 \qquad \text{on } \partial U \, . \end{aligned}$$

   The constant C depends only on λ, U, and on the coefficients of L.

   Explain, in particular, what is meant by "according to the usual energy estimates…"

5. Evans, p. 345, Problem 1

6. Evans, p. 345, Problem 2