1. A connected graph G is called 2-connected if it remains connected after removal of any one of its vertices.

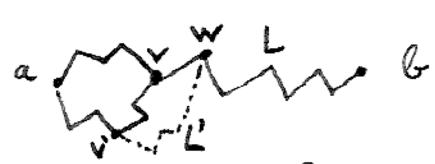
   (a) Give an example of a graph that is connected, but not 2-connected. (5)

   (b) Show that a graph with at least three vertices is 2-connected if and only if every pair of vertices lies in a cycle. (5+5)

a) Any tree with more than two vertices.

b) "$\Rightarrow$": Fix any two distinct vertices a and b.

Since G is 2-connected, a must be part of some cycle C_1. If $b \in C_1$, we are done. Otherwise, we extend C_1 along a simple chain L from a vertex $v \in C_1$ to b (which must exist as G is connected) as follows: Let $w \in L$ be the vertex neighboring C. As G is two-connected, there is a simple chain L' from w to some $v' \in C_1$ avoiding v. Then the concatenation of C_1 without the piece from v to v', the edge (v, w), and L' are again a cycle which now contains w. Continuing, we reach w = b in a finite number of steps along L.



"$\Leftarrow$": Fix $v \in G$ and let $a, b \in G \setminus^2 \{v\}$ be arbitrary. Since there is a cycle containing a and b, there is at least one path between a and b avoiding v.

2. If A and B are points in the plane, let U_{AB} denote the glide reflection along the line AB which maps A to B.

   (a) Given a rectangle with vertices A, B, C, and D, show that

   $$U_{CD} \circ U_{BC} \circ U_{AB} \circ U_{DA} = e.$$

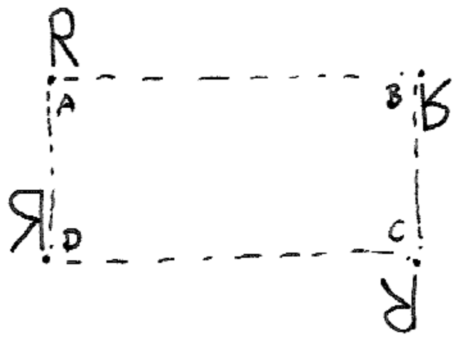
   (In other words, "gliding around the a rectangle" is the identity.)

   (b) Find a condition for a more general quadrilateral that ensures that gliding around it also results in the identity.

   Hint: If the angle between AB and BC is α, by which angle does $U_{BC} \circ U_{AB}$ rotate a vector?

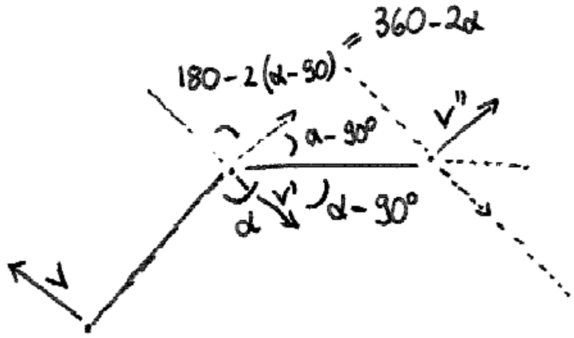
(5+5)

a)



Following the orientation of the letter R as it is mapped around the rectangle proves the claim.

b)



$v' = U_{AB}\, v$

$v'' = U_{BC}\, v'$

So the first two glide reflections rotate a vector by 2α (ignoring the sign), so if β is the angle opposite α, we need $2\alpha + 2\beta = 360°$ or $\alpha + \beta = 180°$.

3. (a) Characterize the group of motions of the line, i.e., the group of maps $\phi\colon \mathbb{R} \to \mathbb{R}$ which preserve the distance between points.

(b) Prove that the set of matrices

$$G = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

is a group with respect to the usual matrix multiplication. Is it Abelian?

(c) Show that the group of motions of the line is isomorphic to G.

Hint: Show that the set

$$L = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

is invariant under G.

(5+5+5)

a) If $y = \phi(x)$, then either $\phi(x+\delta) = y+\delta$ or $\phi(x+\delta) = y-\delta$ as ϕ must preserve distances. In the first case, ϕ is a translation, in the second case it is a composition of translation and reflection.

Thus, the group is generated by all translations and one reflection.

b) By direct computation, we see that

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \lambda \\ 0 & 1 \end{pmatrix}$$

Moreover,

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pm 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pm 1 & \mu+\lambda \\ 0 & 1 \end{pmatrix}$$

and $$\begin{pmatrix} -1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pm 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mp 1 & \lambda-\mu \\ 0 & 1 \end{pmatrix}$$

I.e., G is closed under inversion and matrix multiplikation. Thus, it is a subgroup of the group of invertible 2x2 matrices, hence a group.

c) $$\begin{pmatrix} \pm 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \pm x + \lambda \\ 1 \end{pmatrix}$$

Thus, if we identify $x \in \mathbb{R}$ with $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^2$, then the matrix $\begin{pmatrix} \pm 1 & \lambda \\ 0 & 1 \end{pmatrix}$ corresponds to the motion $\phi(x) = \pm x + \lambda$; by (a) any motion on the line is of this form.

4. A tailor makes jackets and pants. There is enough demand that she sells everything she produces. It takes an hour to make a jacket and half an hour to make a pair of pants. She can spare 10 hours per week for sewing and has a long-running supply contract that provides cloth for 15 pieces per week altogether. The profit on a pair of pants is EUR 15 and the profit on a jacket is EUR 20. How many pieces of each per week should she produce to maximize profit? (10)

Let x denote the number of jackets per week,
y denote the number of pants per week.

Then
$$x + y \leq 15$$
$$x + \tfrac{1}{2}y \leq 10$$

and she ought to maximize $z = 20x + 15y$

In standard form:

$$x + y + s_1 = 15$$
$$x + \tfrac{1}{2}y + s_2 = 10$$
$$x, y, s_1, s_2 \geqslant 0$$

minimize $\zeta = -20x - 15y$.

Let's use the simplex method (graphical is easy as well):

x	y	$\downarrow s_1$	$\downarrow s_2$	
1	1	1	0	15
1	$\frac{1}{2}$	0	1	10
-20	-15	0	0	0

x is entering

s_2 is leaving

$\rightarrow$

$\downarrow x$	y	$\downarrow s_1$	s_2	
0	$\frac{1}{2}$	1	-1	5
1	$\frac{1}{2}$	0	1	10
0	-5	0	20	200

y is entering

s_1 is leaving

$\rightarrow$

x	y	s_1	s_2	
0	1	2	-2	10
1	0	-1	3	5
0	0	10	10	250

the method terminates here.

$\Rightarrow$ She should make $x=5$ jackets and $y=10$ pants at a maximum profit of EUR 250.

5. Suppose each of the following tableaus occurs in the course of performing the simplex algorithm on a linear programming problem in standard form.

(a)

x_1	x_2	x_3	x_4	
0	−1	1	−1	2
1	0	0	2	3
0	−1	0	3	4

x_2 is entering variable, but there is no positive pivot $\Rightarrow$ unbounded optimal solution

(b)

x_1	x_2	x_3	x_4	
0	0	0	1	−1
0	0	1	0	1
2	1	0	0	10

Solution is $x_4 = -1$ (no chance to choose different basic variables)
$\Rightarrow$ feasible region is empty

(c)

x_1	x_2	x_3	x_4	
2	1	0	1	0
1	0	1	4	3
2	0	0	0	8

Basic variable $x_2 = 0$
$\Rightarrow$ degenerate finite optimal solution.

State, for each case, whether

- The feasible region is empty or nonempty;
- The problem has a finite solution;
- if so, whether the solution is degenerate or nondegerate.

(10)

6. Let

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ikjh} v_j$$

denote the discrete Fourier transform of the complex numbers $v_0, \ldots, v_{N+1}$.

Prove the discrete Parseval identity

$$\sum_{k=0}^{N-1} |\tilde{v}_k|^2 = \frac{1}{N} \sum_{j=0}^{N-1} |v_j|^2.$$

(10)

$$\sum_{k=0}^{N-1} |\tilde{v}_k|^2 = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{j=0}^{N-1} e^{-ikhj} v_j \frac{1}{N} \sum_{l=0}^{N-1} e^{iklh} \bar{v}_l$$

$$= \frac{1}{N} \sum_{j,l=0}^{N-1} v_j \bar{v}_l \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} e^{ik(l-j)h}}_{= \delta_{lj}^{per}}$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} |v_j|^2$$

7. Let G be a finite Abelian group of order N, and let χ be a character, i.e., a group homomorphism from G to $\mathbb{C}\setminus\{0\}$. Show that $\chi(a)$ is a root of unity for every $a \in G$. (10)

Since G has order N

$$\chi(a)^N = \chi(Na) = \chi(0) = 1$$

since χ is a group homomorphism and with the convention that we write G additively.

$\Rightarrow$ $\chi(a)$ is an Nth root of unity.