General Mathematics and Computational Science II

Final Exam

May 23, 2014, 12:30–14:30

1. A connected graph G is called 2-connected if it remains connected after removal of any one of its vertices.

   (a) Give an example of a graph that is connected, but not 2-connected. (5)

   (b) Show that a graph with at least three vertices is 2-connected if and only if every pair of vertices lies in a cycle. (5+5)

2. If A and B are points in the plane, let U_{AB} denote the glide reflection along the line AB which maps A to B.

   (a) Given a rectangle with vertices A, B, C, and D, show that

   $$U_{CD} \circ U_{BC} \circ U_{AB} \circ U_{DA} = e.$$

   (In other words, "gliding around the a rectangle" is the identity.)

   (b) Find a condition for a more general quadrilateral that ensures that gliding around it also results in the identity.

   Hint: If the angle between AB and BC is α, by which angle does $U_{BC} \circ U_{AB}$ rotate a vector?

   (5+5)

3. (a) Characterize the group of motions of the line, i.e., the group of maps $\phi \colon \mathbb{R} \to \mathbb{R}$ which preserve the distance between points.

   (b) Prove that the set of matrices

   $$G = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

   is a group with respect to the usual matrix multiplication. Is it Abelian?

(c) Show that the group of motions of the line is isomorphic to G.

Hint: Show that the set

$$L = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

is invariant under G.

(5+5+5)

4. A tailor makes jackets and pants. There is enough demand that she sells everything she produces. It takes an hour to make a jacket and half an hour to make a pair of pants. She can spare 10 hours per week for sewing and has a long-running supply contract that provides cloth for 15 pieces per week altogether. The profit on a pair of pants is EUR 15 and the profit on a jacket is EUR 20. How many pieces of each per week should she produce to maximize profit? (10)

5. Suppose each of the following tableaus occurs in the course of performing the simplex algorithm on a linear programming problem in standard form.

(a)

x_1	x_2	x_3	x_4	
0	−1	1	−1	2
1	0	0	2	3
0	−1	0	3	4

(b)

x_1	x_2	x_3	x_4	
0	0	0	1	−1
0	0	1	0	1
2	1	0	0	10

(c)

x_1	x_2	x_3	x_4	
2	1	0	1	0
1	0	1	4	3
2	0	0	0	8

State, for each case, whether

- The feasible region is empty or nonempty;
- The problem has a finite solution;
- if so, whether the solution is degenerate or nondegerate.

(10)

6. Let

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kjh} \, v_j$$

denote the discrete Fourier transform of the complex numbers $v_0, \ldots, v_{N+1}$.

Prove the discrete Parseval identity

$$\sum_{k=0}^{N-1} |\tilde{v}_k|^2 = \frac{1}{N} \sum_{j=0}^{N-1} |v_j|^2.$$

(10)

7. Let G be a finite Abelian group of order N, and let χ be a character, i.e., a group homomorphism from G to $\mathbb{C}\backslash\{0\}$. Show that $\chi(a)$ is a root of unity for every $a \in G$. (10)