General Mathematics and CPS II

Exercise 23

May 9, 2014

1. Let G be a finite abelian group of order N, written additively, let $\widehat{G}$ denote its dual group, and $\widehat{\widehat{G}}$ the bi-dual group (i.e., the set of characters on $\widehat{G}$).

   (a) Show that for every fixed $a \in G$ the map

   $$\chi \mapsto \chi(a)$$

   defines an element of $\widehat{\widehat{G}}$.

   (b) Show that the construction in part (a) in fact defines a group isomorphism between G and $\widehat{\widehat{G}}$.

2. Let G be a finite abelian group of order N, written additively, and let $\widehat{G}$ denote its dual group. Show that for every nonzero $a \in G$,

   $$\sum_{\chi \in \widehat{G}} \chi(a) = 0\,.$$

   Then conclude that

   $$\sum_{\chi \in \widehat{G}} \overline{\chi(a)}\chi(b) = \begin{cases} N & \text{if } a = b\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   Hint: Mimic the proof of the corresponding orthogonality relation for characters as given in class.

3. Let $f_A \in \mathbb{C}^G$ denote the characteristic function of a set $A \subset G$, i.e.,

   $$f_A(a) = \begin{cases} 1 & \text{if } a \in A\,, \\ 0 & \text{if } a \notin A\,, \end{cases}$$

   and define its *Fourier transform* $\widehat{f}_A \in \mathbb{C}^{\widehat{G}}$ via

   $$\widehat{f}_A(\chi) = \sum_{a \in G} \chi(a)\, f_A(a)\,.$$

Set

$$\Phi(A) = \max\{|\widehat{f}_A(\chi)| \colon \chi \in \widehat{G}, \chi \neq \chi_0\}\,.$$

(Recall that χ_0 denotes the principal character where $\chi_0(a) = 1$ for all $a \in G$.) Show that $\Phi(A) = \Phi(G \setminus A)$.