1. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|z|^2}{4t}}.$$

(a) Show that

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le \frac{C}{t^{n/2}}$$

provided that

$$\|g\|_{L^1} = \int_{\mathbb{R}^n} |g(x)|\, dx < \infty.$$

(b) Suppose w solves the Poisson equation

$$-\Delta w = f \qquad \text{in } \mathbb{R}^n$$

where f is smooth and compactly supported. Show that the solution to the inhomogeneous heat equation

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty),$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

tends to w as $t \to \infty$, i.e., that

$$\lim_{t \to \infty} u(x,t) = w(x)$$

for every $x \in \mathbb{R}^n$.

(10+10)

(a) $$|u(x,t)| \le \int_{\mathbb{R}^n} \Phi(x-y,t)\, |g(y)|\, dy$$

$$\le \underbrace{\max_{z \in \mathbb{R}^n} \Phi(z,t)}_{= \Phi(0,t)} \int_{\mathbb{R}^n} |g(y)|\, dy$$

$$= \frac{1}{t^{n/2}} \frac{\|g\|_{L^1}}{(4\pi)^{n/2}}$$

(b) Since w does not depend on time, $w_t = 0$ and w also solves the inhomogeneous heat equation

Set $\theta = u - w$. Then θ solves the homogeneous heat equation

$$\theta_t - \Delta\theta = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

$$\theta = g - w \quad \text{on } \mathbb{R}^n \times \{t=0\}.$$

Thus, by (a), $\lim_{t\to\infty} |\theta(x,t)| = 0$

provided $g-w \in L^1$, so in particular if both $g, w \in L^1$.

2. Let U be the open unit ball in $\mathbb{R}^n$. Suppose that $u \in C^2_1(\overline{U}_T)$ solves the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } U_T\,, \\ u &= g \qquad \text{on } U \times \{t = 0\}\,, \\ u &= 0 \qquad \text{on } \partial U \times [0, T]\,, \end{aligned} \tag{H}$$

where $U_T = U \times (0, T]$ and $g = 0$ on ∂U.

(a) Show that a radial solution $u(x, t) \equiv v(r, t)$ with $r = |x|$ satisfies

$$\begin{aligned} v_t &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial v}{\partial r}\right) \qquad \text{in } (0, 1) \times (0, T]\,, \\ v &= g \qquad \text{on } (0, 1) \times \{t = 0\}\,, \\ v &= 0 \qquad \text{on } \{r = 1\} \times [0, T]\,. \end{aligned} \tag{R}$$

(b) Show that if $v \in C^2_1([0, 1] \times [0, T])$ solves (R), it is the unique such solution. (In particular, show that a boundary condition at $r = 0$ is not required. Why would you expect this to be so?)

(c) When setting up a numerical discretization scheme for (R), a boundary condition at $r = 0$ might be necessary. What condition could you suggest?

(10+5+5)

(a) $Dv = v'(r)\, Dr = \dfrac{v'(r)}{r}\, x$

$$D \cdot Dv = \left(\frac{v''(r)}{r} - \frac{v'(r)}{r^2}\right) \underbrace{\frac{x}{r} \cdot x}_{= r} + \frac{v'}{r} \underbrace{D \cdot x}_{= n}$$

$$= v''(r) + (n-1)\frac{v'(r)}{r}$$

$$= \frac{1}{r^{n-1}}\left(r^{n-1} v'(r)\right)'$$

This is the radial Laplacian. Restoring the t-dependence of v and inserting into the heat equation, we obtain (R). (The boundary conditions carry over trivially.)

(b) Option 1: Explicit proof

Suppose v and w are solutions. Then $\theta = v - w$ solves

$$\theta_t = \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \theta_r \right) \qquad \text{in } (0,1) \times (0,T]$$

$$\theta = 0 \qquad \text{on } \{r=1\} \times [0,T] \cup (0,1) \times \{t=0\}$$

Now multiply by $r^{n-1}\theta$ and integrate in space:

$$\int_0^1 r^{n-1} \theta \theta_t \, dx = \int_0^1 \theta \, \partial_r \left(r^{n-1} \theta_r \right) dx$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_0^1 \theta^2 r^{n-1} \, dx = \underbrace{\theta \theta_r r^{n-1} \Big|_0^1}_{=0} - \int_0^1 \theta_r^2 r^{n-1} \, dx$$

$=0$ by boundary cond. at $r=1$ and continuity of θ, θ_r up to the boundary.

$$\leq 0$$

Since $\int_0^1 \theta^2 r^{n-1} \, dx$ is a non-negative quantity, zero at $t=0$,

it means it remains zero for all $t \geq 0$.

By a lemma-of-variational-calculus-type argument, this implies that $\theta = 0$ pointwise on $[0,1] \times [0,T]$.

<u>Option 2</u>: A radial solution is a special solution to (H), hence is unique in the larger class of $C_1^2(\overline{U}_T)$ solutions to (H) by the standard uniqueness theorem from class/Evans. The uniqueness claim for (R) is then a strictly weaker statement.

(c) Since $u(x,t) = v(r,t)$, u is only differentiable at the origin if $v_r(0,t) = 0$. This can be used as a computational boundary condition if necessary.

3. Consider the linear transport equation

$$\begin{aligned} u_t + b\,u_x &= 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

We say that $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an integral solution of the transport equation provided

$$\int_0^\infty \int_{\mathbb{R}} u\,(v_t + b\,v_x)\,dx\,dt + \int_{\mathbb{R}} g(x)\,v(x,0)\,dx = 0 \qquad (*)$$

for all $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$.

(a) Suppose that $g \in C(\mathbb{R})$ and that $u \in C^1(\mathbb{R} \times [0,\infty))$ is an integral solution. Show that u solves the transport equation in the classical sense.

(b) Now suppose that $g(x)$ is bounded and smooth except at some $a \in \mathbb{R}$ where it has a jump discontinuity. Show that $u(x,t) = g(x - bt)$ is an integral solution.

(10+10)

(a) We can integrate by parts in (*):

$$-\int_0^\infty \int_{\mathbb{R}} (u_t + b\,u_x)\,v\,dx\,dt + \int_{\mathbb{R}} u v\,dx\,\Big|_{t=0}^{t=\infty} + \int_{\mathbb{R}} g(x)\,v(x,0)\,dx = 0$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}} (u_t + b\,u_x)\,v\,dx\,dt + \int_{\mathbb{R}} (u(x,0) - g(x))\,v(x,0)\,dx = 0 \qquad (**)$$

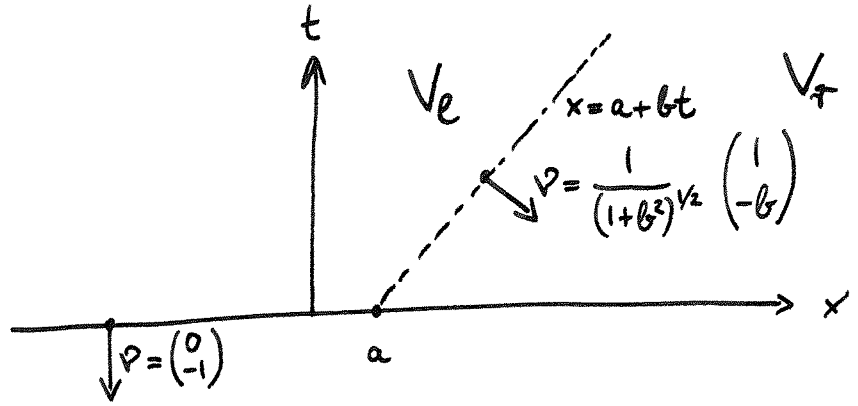
(i) Take $v \in C_c^\infty(\mathbb{R} \times (0,\infty))$ only. Then the second term vanishes and the lemma of variational calculus still applies to the first:

$$u_t + b\,u_x = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty).$$

This extends by continuity to $\mathbb{R} \times [0,\infty)$.

(ii) Thus, the second term in (**) must vanish independently, so that, again by the lemma of variational calculus, $u(x,0) = g(x)$. (And $g \in C^1(\mathbb{R})$ in fact.)

(b) The initial condition is clearly satisfied and for $x - bt \neq a$, $u(x,t)$ satisfies the transport equation in the classical sense, which is easy to check by direct computation.



We write
$$\int_0^\infty \int_{\mathbb{R}} u\,(v_t + b\,v_x)\,dx\,dt = \int_{V_\ell} u \begin{pmatrix} b \\ 1 \end{pmatrix} \cdot Dv\,dx\,dt + \int_{V_r} \%$$

where, using the divergence theorem,
$$\int_{V_\ell} u \begin{pmatrix} b \\ 1 \end{pmatrix} \cdot Dv\,dx\,dt = -\int_{V_\ell} v \underbrace{\begin{pmatrix} b \\ 1 \end{pmatrix} \cdot Du}_{=0 \text{ via classical transport equation}}\,dx\,dt + \int_{\partial V_\ell} v u \begin{pmatrix} b \\ 1 \end{pmatrix} \cdot \nu\,d\ell$$

$$= \int_{\{x = a+bt\}} v u \underbrace{\begin{pmatrix} b \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -b \end{pmatrix}}_{=0} (1+b^2)^{-\frac{1}{2}}\,d\ell + \int_{-\infty}^{a} v(x,0)\,u(x,0) \begin{pmatrix} b \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} dx$$

$$= -\int_{-\infty}^{a} v(x,0)\,g(x)\,dx$$

Similarly $\int_{V_r} u \begin{pmatrix} b \\ 1 \end{pmatrix} \cdot Dv\,dx\,dt = -\int_a^\infty v(x,0)\,g(x)\,dx$

Together, these imply (*), so u is an integral solution.

4. Find the entropy solution for Burgers' equation

$$u_t + u u_x = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$
$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\}.$$

with initial data

$$g(x) = \begin{cases} 1 - x & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

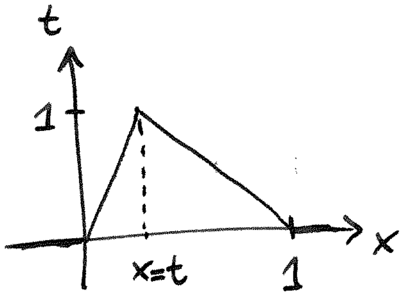
(a) on the interval $t \in (0, 1)$;

(b) for $t \geq 1$.

Hint: Recall that the Rankine–Hugoniot shock condition for a conservation law of the form $u_t + F(u)_x = 0$ states that the solution at an isolated shock curve parameterized by $(s(t), t)$ satisfies $\dot{s}\,[u] = [F(u)]$, the brackets denoting the jump of the enclosed quantity across the shock. (10+10)

(a) By the entropy condition, the initial jump at $t=0$ must resolve into a rarefaction wave. As the maximum at $u=1$ moves with speed 1 and the solution is piecewise linear, it must read

$$u(x,t) = \begin{cases} \frac{x}{t} & \text{for } 0<x<t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



and, in particular, at $t=1$:

$$u(x,1) = \begin{cases} x & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b) When $t \geq 1$, a shock forms which shall be parameterized by $\begin{pmatrix} s(t) \\ t \end{pmatrix}$ with $s(1) = 1$.

By the Rankine-Hugoniot shock condition,

$$\dot{s}\,[u] = \left[\tfrac{1}{2}u^2\right],$$

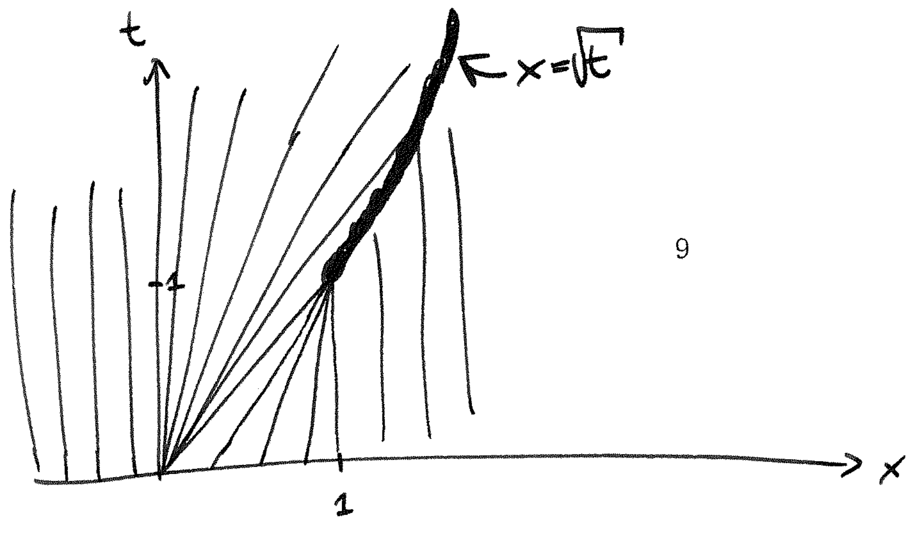
so $$\dot{s}\left(\frac{s}{t} - 0\right) = \frac{1}{2}\left(\frac{s}{t}\right)^2 - \frac{1}{2}0^2$$

$$\Rightarrow \quad \frac{ds}{s} = \frac{1}{2}\frac{dt}{t} \quad \Rightarrow \quad \ln s \Big|_1^{s(t)} = \frac{1}{2}\ln t\Big|_1^t$$

$$\Rightarrow \quad s(t) = \sqrt{t}$$

$$\Rightarrow \quad u(x,t) = \begin{cases} \frac{x}{t} & \text{for } x \in [0, \sqrt{t}] \\ 0 & \text{otherwise} \end{cases}$$

(Not required:) The characteristics in the (x,t) plane are



5. Let $u \in C(\mathbb{R}^3, \mathbb{R}^3)$ be a vector field with

$$|u(x)| \leq \frac{1}{1+|x|^3}.$$

Show that

$$\int_{\mathbb{R}^3} \operatorname{div} u \, dx = 0.$$

(10)

$$\left| \int_{\mathbb{R}^3} \operatorname{div} u \, dx \right| = \left| \lim_{r \to \infty} \int_{B(0,r)} D \cdot u \, dx \right|$$

$$= \left| \lim_{r \to \infty} \int_{\partial B(0,r)} \nu \cdot u \, dx \right|$$

$$\leq \lim_{r \to \infty} \int_{\partial B(0,r)} \underbrace{|\nu|}_{=1} \underbrace{|u|}_{\leq \frac{1}{1+r^3}} dx$$

$$\leq \lim_{r \to \infty} \frac{\pi r^2}{1+r^3} = 0$$

$$\Rightarrow \int_{\mathbb{R}^3} \operatorname{div} u \, dx = 0.$$

6. Let $u(x,t)$ be a smooth solution to the *Korteweg–de Vries equation*

$$u_t - 6uu_x + u_{xxx} = 0$$

on $\mathbb{R} \times (0, \infty)$ such that for every fixed $t \geq 0$, $u(x,t)$ and all its derivatives converge to zero as $|x| \to \infty$.

Show that

$$E(t) = \int_{\mathbb{R}} u(x,t)^2 \, dx$$

remains constant in time. (10)

$$\frac{dE}{dt} = 2 \int_{\mathbb{R}} u u_t \, dx$$

$$= 2 \int_{\mathbb{R}} u(6uu_x - u_{xxx}) \, dx$$

$$= 4 \int_{\mathbb{R}} (u^3)_x \, dx + \underbrace{2 \int_{\mathbb{R}} u_x u_{xx} \, dx}_{= \int_{\mathbb{R}} (u_x^2)_x \, dx}$$

By the fundamental theorem of calculus, since both u and u_x vanish at $\pm\infty$, the two integrals on the RHS are zero.