Introduction to Partial Differential Equations

Homework 8

due May 7, 2013

1. Evans, p. 164, Question 10.

2. (From Evans, p. 164, Question 13.) Let $u \in C(\mathbb{R} \times [0,T])$ for some $T > 0$ be an integral solution to the scalar conservation law

$$\begin{aligned} u_t + F(u)_x &= 0 \qquad \text{in } \mathbb{R} \times (0,T)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

   Assume further that for any fixed $t \in [0,T]$, $u(\,\cdot\,,t)$ has compact support in $\mathbb{R}$, and that $F(0) = 0$. Show that

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} g(x)\,dx$$

   for every $t \in [0,T]$.

3. (From Evans, p. 164, Question 14.) Compute explicitly the unique entropy solution of

$$\begin{aligned} u_t + u\,u_x &= 0 \qquad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

   for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x\,. \end{cases}$$

   Draw the characteristic curves in an (x,t)-plot, being sure to document all qualitative changes in the solution for $t > 0$.