Introduction to Partial Differential Equations

Homework 7

due April 25, 2013

1. Evans, p. 163 problem 3

2. (This question refers to the remark in Evans, p. 114.) Consider the scalar conservation law in one spatial dimension

$$\begin{aligned} u_t + F'(u)\, u_x &= 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \qquad \text{in } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

   (a) Show that the function u defined through the implicit equation

   $$u = g(x - t\, F'(u))$$

   is a solution provided

   $$1 + t\, g'(x - t\, F'(u))\, F''(u) \neq 0\,. \tag{*}$$

   (b) Suppose that $F(u) = \frac{1}{2}\, u^2$. When does solvability condition (*) fail? Is this only a failure of the method of characteristics, or does it also correspond to a failure of the solution to the PDE?

3. Evans, p. 163 problem 5

4. Let L be convex with superlinear growth and suppose that g is uniformly Lipshitz as defined in class (and in Evans, p. 124) via

$$\operatorname{Lip}(g) = \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|} < \infty\,.$$

   Show that if, for fixed $x \in \mathbb{R}^n$ and $t > 0$,

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ t\, L\left(\frac{x - y}{t} \right) + g(y) \right\},$$

   then the infimum in this expression actually is really a minimum.